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L. Bairstow, B. M. Cave and E. D. Lang

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X. *The Resistance of a Cylinder Moving in a Viscous Fluid.*

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Summary of Contents.

Introduction.

An earlier Paper* by the same authors dealt with one aspect of the same problem and in its conclusions indicated the possibility of the present development. In order to deal with the equations of motion for a viscous fluid in two dimensions, the approximation for slow motion due to Stokes was used, with a consequent need for the introduction of a boundary limiting the expanse of the fluid. Whilst keeping the analysis as general as possible, the example given related to a circular cylinder centrally placed in a parallel-walled channel. The extension now to be described follows generally similar lines; the form of equation has been changed from that of Stokes to one proposed by Oseen, the change representing a closer approach to the full equations of motion by the introduction of terms dependent on the inertia of the fluid. A consideration of the differential equations by earlier writers has indicated a close agreement between the motions near a small sphere in the two cases, but a marked difference in the more remote parts of the fluid. Oseen has shown, for the sphere, that the resistance formulæ for the two cases are identical to the first order of small quantities.

In the case of the two-dimensional motion of a cylinder the differences are rather

* 'Roy. Soc. Proc.,' A, vol. 100 (1922).

 [Published, July 20, 1923.

more striking. In the STOKES' form of approximation it is not possible to satisfy all the essential conditions when the expanse of fluid is infinite, whilst with OSEEN'S type of equation this particular difficulty disappears. Having seen OSEEN'S solution for the motion of a sphere in a viscous fluid, LAMB applied a similar method to the circular cylinder, and an account of his analysis is given in the 'Philosophical Magazine,' and his treatise on 'Hydrodynamics' (p. 605); a resistance formula is deduced which is applicable at low velocities. There are two approximations in this solution, one physical and implied in the original differential equation, and the second mathematical and introduced in the solution. There is a certain degree of inter-relation between the two approximations, but it has been found that the second of them may be removed. An estimate of the degree to which OSEEN'S approximation represents the complete equation of viscous fluid motion can be obtained by comparing the results of the new calculations with those of experiment. In the result it appears that the amount still to be accounted for by the remaining inertia terms is less than that already dealt with, in the case of both the resistance of circular cylinders and the skin friction of flat plates.

One of the purposes of the present inquiry was to prepare the ground for a solution of the complete equations of motion for very general boundary forms, and steps are now being taken to that end. The approximate formulæ now dealt with have a validity not anticipated to the extent realised, and appear to give a substantial start for the more complex problem. Both on experimental and mathematical grounds it appears probable that the extension of analysis now contemplated can be sub-divided, and that the remaining inertia terms have an effect of importance whilst the motion remains steady; for higher velocities it is known that the motion behind a cylinder becomes periodic or eddying.

In the present paper resistances have been calculated for cylinders and planes moving through fluid stationary at infinity. In the case of the plane, that case was chosen for which the plane contained the direction of motion. The problem of dealing with a cylinder of any cross-section has been attacked and a method of solution indicated.* In the case of the circular cylinder the process is wholly analytical, for the plate it is mainly so, whilst for a section such as that of an aeroplane wing resort would be made to graphical methods of calculation.

The experiments used for comparison were made at the National Physical Laboratory in the course of the study of Aeronautics. Those on circular cylinders are directly suitable for the present purpose, since the conditions of experiment approached those necessary for two dimensional flow to a satisfactory degree. By a judicious use of the principle of dynamical similarity, the observations provide an internal check on the accuracy of measurement attained.

No corresponding experiments exist for the flat plate, all the reliable experiments having been made on plates which had their greatest dimension along the stream. In

* One of the Research Students of the Imperial College is at present applying the general method to a particular example.

this case the comparison between calculation and observation cannot be precise, and only rough general agreement can be asserted as a result.

The series of experiments on circular cylinders is due to RELF.* Wires of different diameters were tested over a range of speeds in a wind channel. Interpreted in terms of functions suggested by the principles of dynamical similarity, the tests give the value of a resistance coefficient as a function of the product of the speed, diameter of the wire, and the reciprocal of the kinematic coefficient of viscosity. This product is represented in our analysis by a single variable k; and in general it will be found that the type of motion calculated varies with k. At values defined by $\log_{10} 4k = 1 \cdot 0$, $1 \cdot 2$ and $1 \cdot 4$, the calculated resistance co-efficients are 31 per cent., 48 per cent. and 60 per cent. greater than those observed. It is probable that substantial agreement would be obtained between the two values were the experimental range extended to somewhat smaller values of k.

A further report by RELF† on the singing of wires in a wind indicates a limit of k below which singing cannot occur. The report identifies the period of the singing with that of the eddies produced, and places the limit at about $\log_{10} 4k = 2$, *i.e.*, above the range of comparison just quoted. It is, therefore, surmised that the differences now found are not due to unsteady flow in the fluid, and that extension of method may take place by the introduction of further inertia terms and without the complication of periodicity.

Experiments on flat plates have been made by FROUDE, ZAHM, and to a limited extent by Stanton.‡

So far as we are aware the type of extension of method here achieved is new in the published literature of the subject, but we are aware of the work of several authors who are dealing with the motion of viscous fluids. Apart from OSEEN, who stated the equations in the form used in this paper, there are a number of investigators dealing with the slow motion of spheres, both in free space and between walls. In some of these instances, the approximation due to STOKES suffices, as was found in our earlier paper. The particular problem there solved has been independently dealt with by HILDING FAXEN.§ It should, however, be noted that the methods now found are applicable to cylinders of any section, whilst no such generality appears in the analysis by FAXEN. In his earlier paper for his Doctorate, FAXEN gives a bibliography of the subject, and to the references there given we have nothing to add.

The line of attack adopted by us was suggested by LAMB'S|| treatment of the circular

* "Discussion of the Results of Measurements on the Resistance of Wires with some Additional Tests on the Resistance of Wires of Small Diameter." By E. F. RELF, A.R.C.Sc., 'Rept. Advisory Committee for Aeronautics, No. 102,' March, 1914.

† 'Report Aeronautical Research Committee,' T. 1570.

‡ 'Advisory Committee for Aeronautics, R. and M.,' No. 631, February, 1919.

§ 'Annalen der Physik,' vol. 68 (1922).

|| 'Hydrodynamics,' p. 604.

cylinder, and a brief account is consequently given in a form suitable for our development. Certain new functions are introduced, the properties of which involved an appreciable amount of investigation before satisfactory formulæ for calculation were deduced. During the investigation, experience was gained in the graphical manipulation of certain complicated integrals, and it was only at a late stage that the fuller use of analytical methods appeared feasible.

1. LAMB'S *Solution of the Viscous Fluid Motion round a Circular Cylinder.*

In general, the notation used in this paper is identical with that of LAMB'S 'Hydrodynamics' so far as the equations of motion are concerned, and with that of GRAY, MATHEWS and MACROBERT for the various Bessel functions which occur.

ψ being the Lagrangian stream function corresponding with the co-ordinate velocities u and v, LAMB points out* that

$$-\frac{\partial\psi}{\partial y} \equiv u = -\frac{\partial\phi}{\partial x} - c\left(1 - \frac{1}{2k}\frac{\partial}{\partial x}\right) e^{kx}\mathrm{K}_0(kr), \quad \ldots \quad \mathrm{A}\,(1)$$

$$\frac{\partial\psi}{\partial x} \equiv v = -\frac{\partial\phi}{\partial y} + \frac{c}{2k}\frac{\partial}{\partial y} e^{kx}\mathrm{K}_0(kr), \quad \ldots \quad \mathrm{A}\,(2)$$

are expressions which satisfy the equation of viscous fluid motion in the form suggested by OSEEN, *i.e.*,

$$\nu\nabla^2\zeta = \mathrm{U}\frac{\partial\zeta}{\partial x}, \quad \ldots \quad \mathrm{A}\,(3)$$

provided that ϕ satisfies LAPLACE'S equation, K_0 is the BESSEL function of the second kind defined† generally by the relation

$$\mathrm{K}_n(\xi) = \int_0^\infty e^{-\xi\cosh w}\cosh nw\, dw, \quad \ldots \quad \mathrm{A}\,(4)$$

and that $2k = \mathrm{U}/\nu$. c is a general numerical factor, which together with ϕ is to be determined by the boundary conditions.

Introduce two related functions in order to change the form of A (1) and A (2). One of these, ψ_1, is found from ϕ (or instead of ϕ) and is connected with it by the formulæ

$$\frac{\partial\psi_1}{\partial x} = -\frac{\partial\phi}{\partial y}, \qquad \frac{\partial\psi_1}{\partial y} = \frac{\partial\phi}{\partial x}; \quad \ldots \quad \mathrm{A}\,(5)$$

* 'Hydrodynamics,' p. 605.

† GRAY, MATHEWS and MACROBERT, p. 51. 'Treatise on Bessel Functions.'

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it is clearly a harmonic function. The second function, which we have denoted by λ, satisfies the differential relations,

$$\left.\begin{aligned}\frac{\partial\lambda}{\partial x} &= \frac{1}{\pi k}\frac{\partial}{\partial y}e^{kx}\mathrm{K}_0(kr)\\ \frac{\partial\lambda}{\partial y} &= \frac{2}{\pi}\left(1-\frac{1}{2k}\frac{\partial}{\partial x}\right)e^{kx}\mathrm{K}_0(kr)\end{aligned}\right\} \quad \mathrm{A}\,(6)$$

In terms of λ and ψ_1 it is now seen that equations A (1) and A (2) reduce to a single equation

$$\psi = \frac{\pi c}{2}\lambda+\psi_1, \quad \mathrm{A}\,(7)$$

where c and ψ_1 are as yet undetermined.

In an Appendix to this paper it is shown that λ can be expressed in the form of definite integrals as follows :—

$$\pi k\lambda(\xi,\theta) = \xi\mathrm{K}_0(\xi)\int_0^\theta e^{\xi\cos\alpha}\cos\alpha\,d\alpha+\xi\mathrm{K}_1(\xi)\int_0^\theta e^{\xi\cos\alpha}\,d\alpha, \quad \mathrm{A}\,(8)$$

where polar co-ordinates are used which satisfy the relations

$$\xi = kr, \qquad \theta = \tan^{-1} y/x. \quad \mathrm{A}\,(9)$$

When ξ is small, λ can be expanded in the form

$$k\pi\lambda = \theta+\xi\Big[\sin\theta\,\{\mathrm{K}_0(\mathrm{I}_0+\mathrm{I}_2)+2\mathrm{K}_1\mathrm{I}_1\}+\frac{\sin 2\theta}{2}\{\mathrm{K}_0(\mathrm{I}_1+\mathrm{I}_3)+2\mathrm{K}_1\mathrm{I}_2\}$$
$$+\frac{\sin 3\theta}{3}\{\mathrm{K}_0(\mathrm{I}_2+\mathrm{I}_4)+2\mathrm{K}_1\mathrm{I}_3\}+\ldots\Big], \quad \mathrm{A}\,(10)$$

the BESSEL functions K_n and I_n being functions of ξ only.

In following LAMB we retain only the first term of the square bracket and use the expansions for K_0, I_0, etc., retaining only the most important terms, the approximations being

$$\left.\begin{aligned}\mathrm{K}_0 &= -(\gamma+\log\xi/2), &\quad \mathrm{K}_1 &= \frac{1}{\xi},\\ \mathrm{I}_0 &= 1, &\quad \mathrm{I}_1 &= \xi/2.\end{aligned}\right\} \quad \mathrm{A}\,(11)$$

These lead to

$$k\pi\lambda = \theta+\xi\sin\theta\,\{1-\gamma-\log\xi/2\}, \quad \mathrm{A}\,(12)$$

and within the limitations imposed we may take

$$\psi_1 = \alpha\theta+\beta\frac{\sin\theta}{\xi}, \quad \mathrm{A}\,(13)$$

where α and β are constants with respect to ξ and θ. Equation A (7) becomes

$$\psi = \frac{c}{2k}\left\{\theta+\xi\sin\theta\left(1-\gamma-\log\frac{\xi}{2}\right)\right\}+\alpha\theta+\beta\frac{\sin\theta}{\xi}. \quad \mathrm{A}\,(14)$$

The boundary conditions, being those of no slipping relative to the cylinder, may be expressed as

$$\left.\begin{aligned} \overline{\psi}\Big|_b &= \frac{U\xi_0}{k}\sin\theta, \\ \overline{\frac{\partial\psi}{\partial r}}\Big|_b &= k\frac{\partial\psi}{\partial\xi} = U\sin\theta, \end{aligned}\right\} \quad \text{A (15)}$$

where U is the velocity of the cylinder relative to fluid at infinity and ξ_0 is the value of ξ on the boundary. It is easily seen that

$$\alpha = -\frac{c}{2k}, \qquad \beta = -\frac{c}{4k}\xi_0^2, \quad \text{and} \quad c = \frac{2U}{\frac{1}{2}-\gamma-\log\xi_0/2} \quad \text{A (16)}$$

are the correct values to produce agreement between A (14) and A (15), and for small values of kr

$$\psi = \frac{U\sin\theta}{\frac{1}{2}-\gamma-\log kr_0/2}\left\{r\left(1-\gamma-\log\frac{kr}{2}\right)-\frac{r_0^2}{2r}\right\} \quad \text{A (17)}$$

is the final value of the stream function. Corresponding with this LAMB shows that R, the resistance of unit length of the cylinder, is obtained from the formula

$$\frac{R}{\rho U^2 d} = \frac{4\pi\nu/Ud}{\frac{1}{2}-\gamma+\log 8-\log Ud/\nu}, \quad \text{A (18)}$$

d being the diameter of the cylinder.

The value of ψ at points in the fluid for which kr is not small may be obtained from A (7) and the values of c and ψ_1 now found. It differs markedly from that given by A (17) when kr is large. Although λ does not vanish at infinity it may be shown from A (6), by use of the asymptotic expansion for $K_0(kr)$, that its derivatives with respect to x and y are zero. Differentiation of A (7) then shows that the fluid is stationary at infinity, the velocities vanishing to an order not less than $r^{-\frac{1}{2}}$.

Equation A (17) serves as a connecting link between the present paper and our earlier one on "The Two-dimensional Slow Motion of Viscous Fluids."* In the immediate neighbourhood of the cylinder, or more precisely, for small values of REYNOLDS' number Ur/ν it appears from A (17) that the stream function ψ satisfies the differential equation

$$\nabla^4\psi = 0, \quad \text{A (19)}$$

which was given by STOKES as appropriate for slow motion. In our previous enquiry equation A (19) formed the starting point, and the methods there followed go a long way towards establishing equation A (17), which may be re-written as

$$\psi = \frac{c}{2}(1-\gamma-\log k/2)\,r\sin\theta - \frac{c}{2}\sin\theta\left(r\log r+\frac{1}{2}\frac{r_0^2}{r}\right). \quad \text{A (20)}$$

* 'Roy. Soc. Proc.,' A, vol. 100 (1922).

The first term on the right-hand side of A (20) represents a uniform streaming, the effect of which is only to change the velocity of the cylinder through the fluid. Equation A (20) shows a change of uniform streaming from

$$\mathrm{U} \text{ to } -\mathrm{U}\frac{\frac{1}{2}+\log r_0}{\frac{1}{2}-\gamma-\log\frac{kr_0}{2}} \equiv -\frac{c}{2}\left(\tfrac{1}{2}+\log r_0\right). \quad . \quad . \quad . \quad . \quad . \quad \mathrm{A}\,(21)$$

From equation A (19) it follows that the molecular rotation ζ satisfies Laplace's equation, and the appropriate solution for the circular cylinder which satisfies any arbitrary boundary distribution and also vanishes at infinity is

$$\zeta_2 = \mathrm{A}_1\frac{\sin\theta}{r}+\mathrm{A}_2\frac{\sin 2\theta}{r^2}+\ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{A}\,(22)$$

Corresponding with this we have

$$\psi_2 = \frac{\mathrm{A}_1}{2}r\log r\sin\theta-\frac{\mathrm{A}_2}{4}\sin 2\theta+\ldots$$

$$+\mathrm{B}_1\frac{\sin\theta}{r}+\mathrm{B}_2\frac{\sin 2\theta}{r^2}+\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{A}\,(23)$$

where the harmonic function now added also vanishes when r is infinite. The determination of A_1, B_1, etc., to satisfy the boundary velocities given by A (21) presents no difficulty and the final value of ψ_2 is the second term of A (20).

It will be seen therefore that either Stokes' form of equation or that of Oseen will give the details of motion near a cylinder, except for a constant multiplier in the case of the former. This result is closely akin to that found by Oseen in the case of the sphere, where it is found that both equations lead to completely determinate answers. In the immediate neighbourhood of the sphere the solutions agree and lead to a common value for the resistance, but at considerable distances away the motions are very different. Oseen's modification of Stokes' equation is then seen to correspond with a change in the conditions which is fundamental at infinity but becomes unimportant as the moving body is approached.

In our earlier paper it was shown that analysis could be extended in the case of $\nabla^4\psi = 0$ to cover cylinders of non-circular form, and with the freedom given by graphical calculation the section may be chosen arbitrarily. The comparison given above suggests the possibility of similar generality when the equation is

$$\nu\nabla^4\psi = \mathrm{U}\nabla^2\,\partial\psi/\partial x,$$

and the equations are developed at a later stage. (See also p. **424,** Note added April 5, 1923.)

2. *Extension of* LAMB'S *Solution to cover Larger Values of* Ud/ν.

A mathematical limitation other than that in the original differential equation was introduced by LAMB which in the present treatment is represented by A (11). This second approximation is shown by LAMB to be consistent with the first, but when the attempt is made to deal with the unrestricted equations of motion it will be convenient not to have the assumptions of A (11). The problem now solved analytically is otherwise the same as that of LAMB and applies to the circular cylinder only.

The solution already presented may be regarded as founded on the flow of fluid from a doublet, typified by a quantity proportional to λ and situated at the centre of the cylinder. In order to introduce generality into the problem this single doublet is combined with a distributed set of doublets situated on the circular boundary. It will be seen from A (10) that λ is a cyclic function and from A (14) that a cyclic harmonic function has been introduced to eliminate the cyclicity in λ. It is therefore sometimes convenient to use a function L defined by

$$L = \lambda - \theta/k\pi, \qquad \text{B (1)}$$

which is acyclic and which appears to define the two-dimensional motion of a viscous fluid, so far as that is defined by OSEEN'S equation, for points far from a cylinder of any shape.

It is assumed that the Lagrangian stream function appropriate to this problem may be expressed in the form

$$\psi_M = AL_{OM} + \int L_{EM}\, d\chi_E + \sum_1^\infty \frac{G_n \sin nM}{OM^n}, \qquad \text{B (2)}$$

where M is a point in the fluid and E a point lying on the circular boundary, with other quantities as defined in fig. 1. A, $G_1, \ldots G_n \ldots$, are numerical constants, whilst χ_E

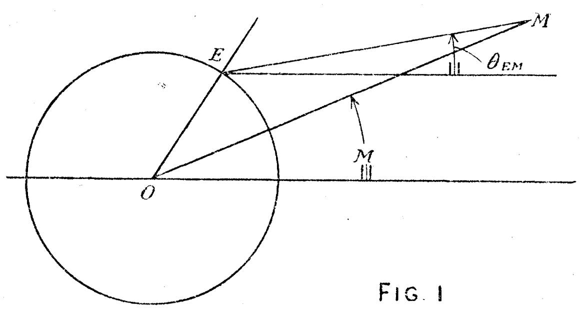


FIG. 1

is a function of the position of E which determines the strength of the distributed doublets.

When M is on the boundary

$$\left.\begin{aligned} \psi_M &= U \cdot OM \cdot \sin M, \\ \frac{\partial\psi}{\partial n_M} &= U \sin M \ (n \text{ radially outwards}). \end{aligned}\right\} \qquad \text{B (3)}$$

From the definition of λ and L given in A (6) and B (1) it will be found that ψ_M satisfies the differential equation

$$\nu\nabla^4\psi = U\nabla^2\,\partial\psi/\partial x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(4)$$

for all values of A, G_n and χ_E, and is therefore a general solution of OSEEN's equation to that extent. These constants and the function χ_E are to be chosen so that the boundary conditions B (3) are satisfied. It needs little analysis to show that the derivatives $\partial\psi/\partial x_M$ and $\partial\psi/\partial y_M$ tend to zero as r tends to infinity, so that ψ_M automatically satisfies the desired boundary conditions there.

OM will be taken as unity since this simplification can be made without any sacrifice of generality.

In dealing with the boundary conditions it is convenient to use differentials of ψ with respect to displacement along and normal to the circle, and these are

$$\frac{\partial\psi}{\partial s_M} = A\frac{\partial L_{OM}}{\partial s_M} + \int \frac{\partial L_{EM}}{\partial s_M}\,d\chi_E + \sum_1^\infty nG_n \cos nM, \quad . \quad . \quad . \quad . \quad . \quad B(5)$$

$$\frac{\partial\psi}{\partial n_M} = A\frac{\partial L_{OM}}{\partial n_M} + \int \frac{\partial L_{EM}}{\partial n_M}\,d\chi_E - \sum_1^\infty nG_n \sin nM. \quad . \quad . \quad . \quad . \quad . \quad B(6)$$

χ_E being a function of E can be expressed in a FOURIER's series, and it is assumed that

$$d\chi_E = \sum_1^\infty C_q \cos qE\,dE\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(7)$$

the form being chosen from considerations of symmetry to be restricted to cosines only.

Using certain theorems in BESSEL functions it has been possible to convert B (5) and B (6) into such form that linear simultaneous algebraic equations remain for the determination of the various coefficients. The number of equations is infinite but the terms converge rapidly and solution is not difficult.

The well-known FOURIER method is used to secure the elimination of G_n between B (5) and B (6). Multiply in B (5) by cos nM and in B (6) by sin nM; add the resulting equations and integrate with respect to M through 2π. The result is to obtain

$$\int_0^{2\pi}\left(\frac{\partial\psi}{\partial s_M}\cos nM + \frac{\partial\psi}{\partial n_M}\sin nM\right)dM = A\int_0^{2\pi}\left(\frac{\partial L_{OM}}{\partial s_M}\cos nM + \frac{\partial L_{OM}}{\partial n_M}\sin nM\right)dM$$
$$+\int_0^{2\pi}\left\{\int\left(\frac{\partial L_{EM}}{\partial s_M}\cos nM + \frac{\partial L_{EM}}{\partial n_M}\sin nM\right)d\chi_E\right\}dM. \quad B(8)$$

In the case of a circular cylinder the values of the derivatives with respect to s and n are readily expressed in terms of those with respect to x and y and

$$\left.\begin{aligned}\frac{\partial}{\partial s_M} &= \cos M\frac{\partial}{\partial y_M} - \sin M\frac{\partial}{\partial x_M}\\ \frac{\partial}{\partial n_M} &= \cos M\frac{\partial}{\partial x_M} + \sin M\frac{\partial}{\partial y_M}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(9)$$

together with the relations given in A (6) and B (1) lead to integrable expressions. In order to shorten the analysis, the process will be indicated in detail in the case of the double integral of B (8) a case which, by simple modification, covers the single integral associated with it.

From B (9) it follows that

$$\cos nM \frac{\partial}{\partial s_M} + \sin nM \frac{\partial}{\partial n_M} = \sin \overline{n-1}M \frac{\partial}{\partial x_M} + \cos \overline{n-1}M \frac{\partial}{\partial y_M}, \quad . \quad . \quad . \quad B(10)$$

whilst from A (6) and A (9) are obtained the relations

$$\frac{\partial L_{EM}}{\partial x_M} = \frac{1}{\pi} e^{kx_{EM}} \frac{\partial K_0(\xi_{EM})}{\partial \xi_{EM}} \sin \theta_{EM} + \frac{1}{\pi} \frac{\sin \theta_{EM}}{\xi_{EM}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad B(11)$$

$$\frac{\partial L_{EM}}{\partial y_M} = \frac{1}{\pi} e^{kx_{EM}} K_0(\xi_{EM}) - \frac{1}{\pi} e^{kx_{EM}} \frac{\partial K_0(\xi_{EM})}{\partial \xi_{EM}} \cos \theta_{EM} - \frac{1}{\pi} \frac{\cos \theta_{EM}}{\xi_{EM}}. \quad . \quad . \quad B(12)$$

The geometry of fig. 1 shows, for M on the boundary and E > M, that

$$\theta_{EM} = 3\frac{\pi}{2} + \frac{E+M}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(13)$$

Hence

$$\begin{aligned} T &\equiv \frac{\partial L_{EM}}{\partial s_M} \cos nM + \frac{\partial L_{EM}}{\partial n_M} \sin nM \\ &= \frac{1}{\pi} e^{kx_{EM}} \left\{ K_0(\xi) \cos \overline{n-1}M - \frac{\partial K_0}{\partial \xi} \cos \frac{E-M}{2} \cdot \sin nM - \frac{\partial K_0}{\partial \xi} \sin \frac{E-M}{2} \cos nM \right\} \\ &\qquad - \frac{1}{2k\pi} \left(\cos nM + \sin nM \cot \frac{E-M}{2} \right). \quad B(14) \end{aligned}$$

The half-angles can be avoided immediately by a change of variable. Since

$$K_0(\xi) = K_0\left(2k \sin \frac{E-M}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(15)$$

it will be seen that

$$\left.\begin{aligned} \frac{\partial K_0}{\partial \xi} \cos \frac{E-M}{2} &= \frac{1}{k} \cdot \frac{\partial K_0(\xi)}{\partial (E-M)} \\ \frac{\partial K_0}{\partial \xi} \sin \frac{E-M}{2} &= \tfrac{1}{2} \frac{\partial K_0(\xi)}{\partial k} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(16)$$

and

On page 74, of Gray, Mathews and MacRobert, Equation 59, an expansion is given for K_0 (ξ) in the form

$$K_0(\xi) = I_0(k) K_0(k) + 2 \sum_{p=1}^{\infty} I_p(k) K_p(k) \cos p(E-M), \quad . \quad . \quad . \quad B(17)$$

and is valid as quoted, except that care is needed when E = M.

By differentiation of B (17)

$$\frac{1}{k}\cdot\frac{\partial K_0(\xi)}{\partial(E-M)} = -\sum_{p=1}^{\infty}\frac{2p}{k} I_pK_p \sin p(E-M)$$

$$= -\sum_{p=1}^{\infty} K_p(I_{p-1}-I_{p+1}) \sin p(E-M) \quad . \quad . \quad . \quad . \quad B(18)$$

and

$$\tfrac{1}{2}\cdot\frac{\partial K_0(\xi)}{\partial k} = I_1K_0 - \frac{1}{2k} + \sum_{p=1}^{\infty}\left\{K_p(I_{p-1}+I_{p+1}) - \frac{1}{k}\right\} \cos p(E-M) \quad . \quad . \quad B(19)$$

where use has been made of the relation

$$K_{p+1}I_p + K_pI_{p+1} = \frac{1}{k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad B(20)$$

Inserting these relations B (14) becomes

$$T = \frac{1}{\pi}e^{kx_{EM}}\Big[I_0K_0 \cos \overline{n-1}\,M + \left(\frac{1}{2k} - I_1K_0\right)\cos nM + 2\cos\overline{n-1}\,M \sum_{p=1}^{\infty} I_pK_p \cos p(E-M)$$

$$+ \sin nM \sum_{p=1}^{\infty} K_p(I_{p-1}-I_{p+1}) \sin p(E-M)$$

$$- \cos nM \sum_{p=1}^{\infty}\left\{K_p(I_{p-1}+I_{p+1}) - \frac{1}{k}\right\} \cos p(E-M)\Big]$$

$$- \frac{1}{2k\pi}\left(\cos nM + \sin nM \cot\frac{E-M}{2}\right). \quad . \quad . \quad B(21)$$

After a number of straightforward trigonometrical transformations it can be seen that

$$T = \frac{1}{2\pi}e^{kx_{EM}}\left\{{}_n\beta_0 + 2\sum_{p=1}^{\infty}({}_n\beta_p \cos pE + {}_n\gamma_p \sin pE)\right\}$$

$$- \frac{1}{2k\pi}\left\{\cos nM + \sin nM \cot\frac{E-M}{2}\right\}. \quad . \quad . \quad B(22)$$

where

$${}_n\beta_p = I_pK_p(\cos\overline{n-1-p}\,M + \cos\overline{n-1+p}\,M)$$

$$+ \left(\frac{1}{2k} - K_pI_{p-1}\right)\cos\overline{n-p}\,M + \left(\frac{1}{2k} - K_pI_{p+1}\right)\cos\overline{n+p}\,M, \quad . \quad . \quad B(23)$$

$${}_n\gamma_p = I_pK_p(-\sin\overline{n-1-p}\,M + \sin\overline{n-1+p}\,M)$$

$$- \left(\frac{1}{2k} - K_pI_{p-1}\right)\sin\overline{n-p}\,M + \left(\frac{1}{2k} - K_pI_{p+1}\right)\sin\overline{n+p}\,M. \quad . \quad . \quad B(24)$$

Since the radius of the cylinder has been taken as unity the value of

$$x_{EM} \equiv x_M - x_E \quad \text{is equal to} \quad \cos M - \cos E \quad . \quad . \quad . \quad . \quad . \quad B(25)$$

and the exponential factor of B (22) is the product of a function of E and a function of M.

The next step is the integration of T with respect to χ_E using, for the latter quantity, the FOURIER expansion which was postulated in B (7). The integral required is

$$\int T\,d\chi_E = \frac{1}{2\pi} e^{k\cos M} \int e^{-k\cos E} \{{}_n\beta_0 + 2\sum_{p=1}^{\infty} ({}_n\beta_p \cos pE + {}_n\gamma_p \sin pE)\}\, d\chi_E$$
$$- \frac{\sin nM}{2k\pi} \int \cot\frac{E-M}{2}\, d\chi_E. \quad \text{B (26)}$$

Equation B (7) being $d\chi_E = \sum_1^{\infty} C_q \cos qE\, dE$ it would be possible to integrate in B (26) by the use of this expression and the standard theorems

$$\int_M^{2\pi+M} e^{-k\cos E} \cos sE\, dE = 2\pi(-1)^s I_s. \quad \text{B (27)}$$

and

$$\int_M^{2\pi+M} e^{-k\cos E} \sin sE\, dE = 0. \quad \text{B (28)}$$

It is, however, preferable to make a different assumption to B (7) and to write

$$e^{-k\cos E}\, d\chi_E = \sum_0^{\infty} a_q \cos qE\, dE. \quad \text{B (29)}$$

Expanding the exponential function in BESSEL functions and multiple cosines leads to the relation

$$q \neq 0 \qquad a_q = \sum_1^{\infty} (-1)^{q+r} C_r (I_{q+r} + I_{q-r}). \quad \text{B (30)}$$

and

$$a_0 = \sum_1^{\infty} (-1)^r C_r I_r.$$

Multiplying both sides of B (29) by $e^{k\cos E}$ and again expanding gives the reciprocal relation

$$C_q = \sum_0^{\infty} a_r (I_{q+r} + I_{q-r}). \quad \text{B (31)}$$

It was pointed out at an earlier stage that C_0 was zero and hence from B (31) we must have

$$0 = \sum_0^{\infty} a_r I_r \quad \text{B (32)}$$

Equation B (26) now becomes

$$\int T\,d\chi_E = \frac{1}{2\pi} e^{k\cos M} \sum_0^{\infty} a_q \int_M^{2\pi+M} \{{}_n\beta_0 + 2\sum_{p=1}^{\infty} ({}_n\beta_p \cos pE + {}_n\gamma_p \sin pE)\} \cos qE\, dE$$
$$- \frac{\sin nM}{2k\pi} \sum_0^{\infty} a_q \int_M^{2\pi+M} \cot\frac{E-M}{2} \{I_0 + 2\sum_{p=1}^{\infty} I_p \cos pE\} \cos qE\, dE, \quad \text{B (33)}$$
$$= e^{k\cos M} \sum_0^{\infty} a_q\, {}_n\beta_q + \frac{\sin nM}{k} \sum_0^{\infty} a_q \{I_0 \sin qM + \sum_{p=1}^{\infty} I_p (\sin \overline{p+q}\, M + \sin \overline{p-q}\, M)\},$$
$$\quad \text{B (34)}$$

where use has been made of the theorem

$$\int_{M}^{2\pi+M} \cot\frac{E-M}{2}\cos sE\,dE = -2\pi\sin sM. \qquad B(35)$$

Using B (28) and B (29) to assist in the integration it is readily found that

$$\int_0^{2\pi}\left\{\int T\,d\chi_E\right\}dM = 2\pi\sum_0^\infty a_q I_q\,\{K_q(I_{n-1-q}+I_{n-1+q})+K_{q-1}I_{n-q}+K_{q+1}I_{n+q}\}. \qquad B(36)$$

The remaining terms of B (8) present little difficulty; from B (14) we have

$$\int_0^{2\pi}\left\{\frac{\partial L_{OM}}{\partial s_M}\cos nM+\frac{\partial L_{OM}}{\partial n_M}\sin nM\right\}dM$$

$$= -\frac{1}{k\pi}\int_0^{2\pi}\cos nM\,dM+\frac{1}{\pi}\int_0^{2\pi}e^{k\cos M}\{K_0\cos\overline{n-1}\,M+K_1\cos nM\}\,dM$$

$$= 2\{K_0I_{n-1}+K_1I_n\}, \qquad B(37)$$

The remaining integral of B (8) is

$$\int_0^{2\pi}\left\{\frac{\partial\psi}{\partial s_M}\cos nM+\frac{\partial\psi}{\partial n_M}\sin nM\right\}dM = \int_0^{2\pi}U\cos\overline{n-1}\,M\,dM$$

$$= 0 \text{ except when } n = 1$$

$$= 2\pi U \text{ when } n = 1. \qquad B(38)$$

The whole of equation B (8) now becomes

$$\left.\begin{matrix}U\,(n=1)\\ 0\,(n>1)\end{matrix}\right\} = \frac{A}{\pi}(K_0I_{n-1}+K_1I_n)+\sum_0^\infty a_qI_q\,\{K_q(I_{n-1-q}+I_{n-1+q})+K_{q-1}I_{n-q}+K_{q+1}I_{n+q}\}. \qquad B(39)$$

When using B (39) for numerical calculation to determine a_q it appeared the summation covered values of which the greater part vanished in virtue of B (32). The equation may therefore be usefully re-written as

$$\left.\begin{matrix}U\,(n=1)\\ 0\,(n>1)\end{matrix}\right\} = \frac{A}{\pi}(K_0I_{n-1}+K_1I_n)+\sum_1^\infty a_qI_q\,\{K_q(I_{n-1-q}+I_{n-1+q})+K_{q-1}I_{n-q}+K_{q+1}I_{n+q}$$
$$-2\,(K_0I_{n-1}+K_1I_n)\}. \qquad B(40)$$

For each value of n equation B (40) gives a linear relation between the coefficients A, a_1, a_2 ... a_q. It was assumed that a limited number of equations would suffice for their determination and the results have left no doubt as to the adequacy of this approximate method of solution.

A number of numerical applications have been made using the tables provided by Gray, Mathews and MacRoberts for the Bessel functions I_n and K_n. Throughout the calculations the radius of the cylinder has been taken as unity; generalization is readily obtained by the use of the principle of dynamical similarity.

The variable in problems on viscous fluid motion of cylinders which has the greatest physical significance is Ud/ν and in terms of our analysis, where $d = 2$ and $U = 1$ this quantity is equal to $4k$.

TABLE I.

k	0	0·05	0·2	1	2	3	5
C_1 C_2 C_3 C_4 C_5 C_6	0 *	−0·047 *	−0·185 *	−1·00 −0·05 *	−2·11 −0·28 −0·12 *	−3·19 −0·63 −0·36 −0·13 *	−5·22 −1·33 −1·04 −0·58 −0·22 −0·10
A	∞	0·8663	1·356	2·921	4·48	5·94	8·70
Resistance Coefficient.	∞	17·33	6·78	2·92	2·24	1·98	1·74
Ud/ν	0	0·2	0·8	4	8	12	20

The convergence of χ_E as exhibited by the coefficients C_q of Table I is very great for small values of k, and is still satisfactorily rapid for values of k which render it possible to make a direct comparison with experiment where the smallest value of Ud/ν reached corresponds with $k = 3$. The agreement with LAMB's form for small values of k is shown by the vanishing of C_q and the agreement of A with the value found by him.

3. *Evaluation of u, v and ψ.*

The integration already carried out for the circular cylinder when M is on the boundary can be extended to cover any general point. Equation B (2) is

$$\psi_M = AL_{OM} + \int L_{EM}\, d\chi_E + \sum_1^\infty G_n \frac{\sin nM}{OM^n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad C\,(1)$$

and by direct differentiation, remembering that

$$\left.\begin{aligned} \frac{\partial}{\partial x_M} &= \cos M \frac{\partial}{\partial OM} - \frac{\sin M}{OM} \cdot \frac{\partial}{\partial M} \\ \text{and} \qquad \frac{\partial}{\partial y_M} &= \sin M \frac{\partial}{\partial OM} + \frac{\cos M}{OM} \cdot \frac{\partial}{\partial M}, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad C\,(2)$$

we get

$$-u_M \equiv \frac{\partial \psi}{\partial y_M} = A \frac{\partial L_{OM}}{\partial y_M} + \int \frac{\partial L_{EM}}{\partial y_M}\, d\chi_E + \sum_1^\infty G_n \frac{n \cos \overline{n+1}\, M}{OM^{n+1}} \quad . \quad . \quad . \quad C\,(3)$$

and

$$v_{\mathrm{M}} \equiv \frac{\partial \psi}{\partial x_{\mathrm{M}}} = \mathrm{A}\,\frac{\partial \mathrm{L}_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} + \int \frac{\partial \mathrm{L}_{\mathrm{EM}}}{\partial x_{\mathrm{M}}}\,d\chi_{\mathrm{E}} - \sum_{1}^{\infty} \mathrm{G}_n\,\frac{n \sin \overline{n+1}\,\mathrm{M}}{\mathrm{OM}^{n+1}} \quad . \quad . \quad . \quad \mathrm{C}\,(4)$$

The functions $\partial \mathrm{L}_{\mathrm{OM}}/\partial y_{\mathrm{M}}$, etc., are known from the general definition of λ and L and the only new step is the integration for χ_{E}. Equations C (3) and C (4) become

$$-u_{\mathrm{M}} = \mathrm{A}\,\frac{\partial \mathrm{L}_{\mathrm{OM}}}{\partial y_{\mathrm{M}}} + \frac{2}{\pi}\left(1 - \frac{1}{2k}\,\frac{\partial}{\partial x_{\mathrm{M}}}\right)\int e^{kx_{\mathrm{EM}}}\,.\,\mathrm{K}_0\,(k\mathrm{EM})\,d\chi_{\mathrm{E}}$$
$$- \frac{1}{k\pi}\,\frac{\partial}{\partial x_{\mathrm{M}}}\int \log k\mathrm{EM}\,d\chi_{\mathrm{E}} + \sum_{1}^{\infty} \mathrm{G}_n\,\frac{n \cos \overline{n+1}\,\mathrm{M}}{\mathrm{OM}^{n+1}} \quad . \quad . \quad . \quad \mathrm{C}\,(5)$$

and

$$v_{\mathrm{M}} = \mathrm{A}\,\frac{\partial \mathrm{L}_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} + \frac{1}{k\pi}\,\frac{\partial}{\partial y_{\mathrm{M}}}\int e^{kx_{\mathrm{EM}}}\mathrm{K}_0\,(k\mathrm{EM})\,d\chi_{\mathrm{E}}$$
$$+ \frac{1}{k\pi}\,\frac{\partial}{\partial y_{\mathrm{M}}}\int \log k\mathrm{EM}\,d\chi_{\mathrm{E}} - \sum_{1}^{\infty} \mathrm{G}_n\,\frac{n \sin \overline{n+1}\,\mathrm{M}}{\mathrm{OM}^{n+1}} \quad . \quad . \quad . \quad \mathrm{C}\,(6)$$

In both cases the values of the integrals

$$\int e^{kx_{\mathrm{EM}}}\,\mathrm{K}_0\,(k\mathrm{EM})\,d\chi_{\mathrm{E}} \quad \text{and} \quad \int \log k\mathrm{EM}\,d\chi_{\mathrm{E}}$$

are required and it is desirable to estimate these separately.

Taking the radius of the cylinder as unity and using the notation $\xi = k\,.\,\mathrm{OM}$ we have as a general theorem in Bessel functions

$$\mathrm{K}_0\,(k\mathrm{EM}) = \mathrm{I}_0\,(k)\,.\,\mathrm{K}_0\,(\xi) + 2\sum_{p=1}^{\infty} \mathrm{I}_p\,(k)\,\mathrm{K}_p\,(\xi)\cos p\,(\mathrm{E}-\mathrm{M}), \quad . \quad . \quad . \quad \mathrm{C}\,(7)$$

and therefore

$$\int e^{kx_{\mathrm{EM}}}\,\mathrm{K}_0\,(k\mathrm{EM})\,d\chi_{\mathrm{E}} = \int e^{\xi \cos \mathrm{M} - k \cos \mathrm{E}}\left\{\mathrm{I}_0\,(k)\,\mathrm{K}_0\,(\xi) + 2\sum_{p=1}^{\infty} \mathrm{I}_p\,(k)\,\mathrm{K}_p\,(\xi)\cos p\,(\mathrm{E}-\mathrm{M})\right\} d\chi_{\mathrm{E}} \quad . \quad . \quad . \quad . \quad . \quad \mathrm{C}\,(8)$$

$$= \sum_{0}^{\infty} a_q e^{\xi \cos \mathrm{M}} \int_{\mathrm{M}}^{\mathrm{M}+2\pi} \left\{\mathrm{I}_0\,(k)\,\mathrm{K}_0\,(\xi) + 2\sum_{p=1}^{\infty} \mathrm{I}_p\,(k)\,\mathrm{K}_p\,(\xi)\cos p\,\overline{\mathrm{E}-\mathrm{M}}\right\} \cos q\mathrm{E}\,d\mathrm{E}$$

$$= e^{\xi \cos \mathrm{M}}\left\{2\pi a_0 \mathrm{I}_0\,(k)\,\mathrm{K}_0\,(\xi) + \sum_{q=1}^{\infty} a_q \int_0^{2\pi} \sum_{p=1}^{\infty} \mathrm{I}_p\,(k)\,\mathrm{K}_p\,(\xi)\cos p\,\overline{\mathrm{E}-\mathrm{M}}\cos q\mathrm{E}\,d\mathrm{E}\right\},$$

and finally

$$\int e^{kx_{\mathrm{EM}}}\,\mathrm{K}_0\,(k\mathrm{EM})\,d\chi_{\mathrm{E}} = 2\pi e^{\xi \cos \mathrm{M}} \sum_{0}^{\infty} a_q \mathrm{I}_q\,(k)\,\mathrm{K}_q\,(\xi)\cos q\mathrm{M}. \quad . \quad . \quad . \quad \mathrm{C}\,(9)$$

Using the trigonometrical expansion

$$\log k\mathrm{EM} = \log k\mathrm{OM} - \sum_{1}^{\infty} \frac{1}{n}\,\frac{\cos n\,\overline{\mathrm{E}-\mathrm{M}}}{\mathrm{OM}^n}$$

leads to the evaluation of $\int \log k\mathrm{EM}\, d\chi_{\mathrm{E}}$, for

$$\int \log k\mathrm{EM}\, d\chi_{\mathrm{E}} = -\sum_1^{\infty}\frac{1}{n}\cdot\frac{1}{\mathrm{OM}^n}\int \cos n\,\overline{\mathrm{E}-\mathrm{M}}\, d\chi_{\mathrm{E}}$$

$$= -\sum_1^{\infty}\frac{1}{n}\cdot\frac{1}{\mathrm{OM}^n}\int_{\mathrm{M}}^{\mathrm{M}+2\pi} e^{k\cos\mathrm{E}}\cos n\,\overline{\mathrm{E}-\mathrm{M}}\sum_{q=0}^{\infty} a_q\cos q\mathrm{E}\, d\mathrm{E}$$

$$= -\pi\sum_{n=1}^{\infty}\frac{\cos n\mathrm{M}}{n\,\mathrm{OM}^n}\sum_{q=0}^{\infty} a_q(\mathrm{I}_{n+q}+\mathrm{I}_{n-q}). \quad \mathrm{C}\,(10)$$

The relations so found lead to the essential elements of the component velocities u and v.

$$-u = \mathrm{A}\frac{\partial \mathrm{L}_{\mathrm{OM}}}{\partial y_{\mathrm{M}}} + 4\left(1-\frac{1}{2k}\frac{\partial}{\partial x_{\mathrm{M}}}\right)e^{\xi\cos\mathrm{M}}\sum_0^{\infty} a_q\mathrm{I}_q(k)\,\mathrm{K}_q(\xi)\cos q\mathrm{M}$$

$$+\sum_1^{\infty}\frac{\cos\overline{n+1}\,\mathrm{M}}{\mathrm{OM}^{n+1}}\left\{n\mathrm{G}_n-\frac{1}{k}\sum_{q=0}^{\infty} a_q(\mathrm{I}_{n+q}+\mathrm{I}_{n-q})\right\} \quad \mathrm{C}\,(11)$$

and

$$v = \mathrm{A}\frac{\partial \mathrm{L}_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} + \frac{2}{k}\cdot\frac{\partial}{\partial y_{\mathrm{M}}}e^{\xi\cos\mathrm{M}}\sum_0^{\infty} a_q\mathrm{I}_q(k)\,\mathrm{K}_q(\xi)\cos q\mathrm{M}$$

$$-\sum_1^{\infty}\frac{\sin\overline{n+1}\,\mathrm{M}}{\mathrm{OM}^{n+1}}\left\{n\mathrm{G}_n-\frac{1}{k}\sum_{q=0}^{\infty} a_q(\mathrm{I}_{n+q}+\mathrm{I}_{n-q})\right\}. \quad \mathrm{C}\,(12)$$

The differentiations present no difficulties. Changing to polar co-ordinates for part of the integration gives ψ as

$$\psi_{\mathrm{M}} = \mathrm{AL}_{\mathrm{OM}} + 2\xi\sum_0^{\infty} a_q\mathrm{I}_q(k)\left\{\mathrm{K}_q(\xi)\int_0^{\mathrm{M}} e^{\xi\cos\alpha}\cos\alpha\cos q\alpha\, d\alpha - \mathrm{K}'_q(\xi)\int_0^{\mathrm{M}} e^{\xi\cos\alpha}\cos q\alpha\, d\alpha\right\}$$

$$+\sum_1^{\infty}\frac{\sin n\mathrm{M}}{n\,.\,\mathrm{OM}^n}\left\{n\mathrm{G}_n-\frac{1}{k}\sum_{q=0}^{\infty} a_q(\mathrm{I}_{n+q}+\mathrm{I}_{n-q})\right\} \quad \mathrm{C}\,(13)$$

and evaluation depends on the possibility of finding certain definite integrals. For the case in which $q = 0$ both may be evaluated by a method given in an Appendix to this paper. It will be shown shortly that a recurrence formula can be developed which makes computation for $q > 0$ depend on the integrals already dealt with. Before doing this, however, it should be noted that all are cyclic, although this property does not apply to the sum involved in ψ_{M}. This may be seen as follows: the cyclic terms in ψ_{M} are

$$2\xi\mathrm{M}\sum_0^{\infty} a_q\mathrm{I}_q(k)\left\{\mathrm{K}_q(\mathrm{I}_{q+1}+\mathrm{I}_{q-1})-2\mathrm{K}'_q\mathrm{I}_q\right\}$$

$$= 2\xi\mathrm{M}\sum_0^{\infty} a_q\mathrm{I}_q(k)\left\{\mathrm{K}_q(\mathrm{I}_{q+1}+\mathrm{I}_{q-1})+(\mathrm{K}_{q+1}+\mathrm{K}_{q-1})\,\mathrm{I}_q\right\} \quad \mathrm{C}\,(14)$$

$$= 4\mathrm{M}\sum_0^{\infty} a_q\mathrm{I}_q(k) = 0. \quad \mathrm{C}\,(15)$$

The relationship

$$\mathrm{I}_q\mathrm{K}_{q+1}+\mathrm{I}_{q+1}\mathrm{K}_q = \frac{1}{\xi}$$

has here been used to convert C (14) to C (15) whilst the sum is zero by B (32).

It is advantageous to work with acyclic functions, and it is therefore suggested that a new function be defined as

$$\tau_p = \int_0^{\mathrm{M}} e^{\xi \cos \alpha} \cos p\alpha \, d\alpha - \mathrm{MI}_p(\xi), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{C}\,(16)$$

from which we obtain the relationship

$$\begin{aligned} \tau_{p+1} - \tau_{p-1} &= \int_0^{\mathrm{M}} e^{\xi \cos \alpha} (\cos \overline{p+1}\, \alpha - \cos \overline{p-1}\, \alpha) \, d\alpha - \mathrm{M}\,(\mathrm{I}_{p+1} - \mathrm{I}_{p-1}) \\ &= -2 \int_0^{\mathrm{M}} e^{\xi \cos \alpha} \sin \alpha \sin p\alpha \, d\alpha + \mathrm{M}\,\frac{2p}{\xi}\,\mathrm{I}_p \\ &= \frac{2}{\xi} \Big[e^{\xi \cos \alpha} \sin p\alpha \Big]_0^{\mathrm{M}} - \frac{2p}{\xi} \int_0^{\mathrm{M}} e^{\xi \cos \alpha} \cos p\alpha \, d\alpha + \mathrm{M}\,\frac{2p}{\xi}\,\mathrm{I}_p \\ &= \frac{2}{\xi} \{ e^{\xi \cos \mathrm{M}} \sin p\mathrm{M} - p\tau_p \}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{C}\,(17) \end{aligned}$$

τ_0 and τ_1 may be found by methods described in the Appendix for both large and small values of ξ; equation C (17) then gives the rest of the functions necessary for the evaluation of ψ.

In order to complete the solution for ψ it is convenient to find G_n from C (13), by using the known boundary value of $\partial\psi/\partial\mathrm{M}$.

Since

$$k\pi\mathrm{L}\,(\xi, \mathrm{M}) = -\mathrm{M} + \xi \int_0^{\mathrm{M}} e^{\xi \cos \alpha} \{ \mathrm{K}_0(\xi) \cos \alpha - \mathrm{K}'_0(\xi) \} \, d\alpha, \quad . \quad . \quad . \qquad \mathrm{C}\,(18)$$

we have

$$\begin{aligned} \frac{\partial\psi}{\partial\mathrm{M}} = -\frac{\mathrm{A}}{k\pi} + \frac{\mathrm{A}\xi}{k\pi} e^{\xi \cos \mathrm{M}} \{ \mathrm{K}_0(\xi) \cos \mathrm{M} - \mathrm{K}'_0(\xi) \} \\ + 2\xi e^{\xi \cos \mathrm{M}} \sum_0^{\infty} a_q \mathrm{I}_q(k) \cos q\mathrm{M} \{ \mathrm{K}_q(\xi) \cos \mathrm{M} - \mathrm{K}'_q(\xi) \} \\ + \sum_1^{\infty} \frac{\cos n\mathrm{M}}{\mathrm{OM}^n} \left\{ n\mathrm{G}_n - \frac{1}{k} \sum_{q=0}^{\infty} a_q (\mathrm{I}_{n+q} + \mathrm{I}_{n-q}) \right\}. \quad . \qquad \mathrm{C}\,(19) \end{aligned}$$

When $\xi = k$, the value of $\dfrac{\partial\psi}{\partial\mathrm{M}}$ is U cos M, and by the use of SONINE's expansion for $e^{\xi \cos \mathrm{M}}$ the values of G_n may be found from C (19).

4. *Calculation of Resistance.*

The formulæ for component pressures as deduced by STOKES can be found in LAMB's 'Hydrodynamics,' p. 570, of which the two relevant to the present problem are

$$\begin{aligned} p_{xx} &= -p + 2\mu \frac{\partial u}{\partial x} \\ p_{xy} &= \mu \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{D}\,(1) \end{aligned}$$

The force on the element of surface at M, fig. 2, in the direction of the axis of x is pro-

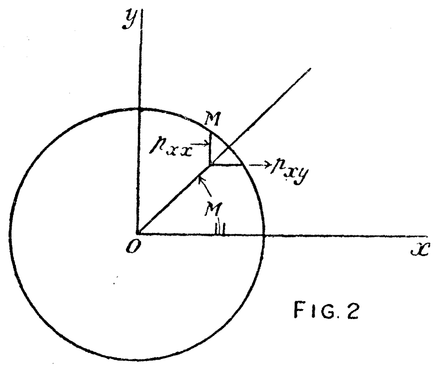


FIG. 2

duced by the pressure p_{xx} over the vertical facet and the shear stress p_{xy} on the horizontal facet. The resistance per unit length of the cylinder is

$$R = \int_0^{2\pi} p_{xx} \cos M \, dM + \int_0^{2\pi} p_{xy} \sin M \, dM, \quad \text{D (2)}$$

$$= \int_0^{2\pi} \left\{ -p \cos M + 2\mu \frac{\partial u}{\partial x} \cos M + \mu \sin M \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right\} dM. \quad \text{D (3)}$$

This expression can be simplified by making use of the boundary conditions, for on the surface of the cylinder u is constant and v zero.

Hence

$$\frac{\partial u}{\partial s} = 0 = \cos M \frac{\partial u}{\partial y} - \sin M \frac{\partial u}{\partial x}, \quad \text{D (4)}$$

and

$$\frac{\partial v}{\partial s} = 0 = \cos M \frac{\partial v}{\partial y} - \sin M \frac{\partial v}{\partial x}, \quad \text{D (5)}$$

and from these it follows that

$$\cos^2 M \frac{\partial u}{\partial y} + \sin^2 M \frac{\partial v}{\partial} = \sin M \cos M \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad \text{D (6)}$$

The term on the right-hand side of D (6) is zero by virtue of the equation of continuity, and therefore

$$\cos^2 M \frac{\partial u}{\partial y} + \sin^2 M \frac{\partial v}{\partial x} = 0. \quad \text{D (7)}$$

Since $\zeta = \partial v/\partial x - \partial u/\partial y$ by definition, it can be seen that $\partial v/\partial x$ and $\partial u/\partial y$ can be obtained in terms of ζ and M by use of D (4) and D (7) to get

$$\left. \begin{aligned} \partial u/\partial y &= -\zeta \sin^2 M, \\ \partial u/\partial x &= -\zeta \sin M \cos M. \end{aligned} \right\} \quad \text{D (8)}$$

From D (8) and the equation for ζ it will be found that D (3) reduces to

$$R = \mu \int_0^{2\pi} \left\{ -\frac{p}{\mu} \cos M - \zeta \sin M \right\} dM. \quad \ldots \ldots \quad D\,(9)$$

The value of ζ is readily found from that of ψ by differentiation, but it remains that p is still to be found from the equations of motion.

In the form consistent with OSEEN'S approximation the equations containing p are*

$$\rho U \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} - \mu \frac{\partial \zeta}{\partial y}, \quad \ldots \ldots \quad D\,(10)$$

$$\rho U \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \mu \frac{\partial \zeta}{\partial x}, \quad \ldots \ldots \quad D\,(11)$$

and the solution to this point has been obtained from a combination of D (10) and D (11) in which p does not appear. Differentiation of D (10) with respect to x and of D (11) with respect to y leads by addition, and the use of the equation of continuity to

$$\nabla^2 (p) = 0, \quad \ldots \ldots \quad D\,(12)$$

and p is a harmonic function. Equations D (10) and D (11) may be re-written as

$$\left.\begin{aligned} -\frac{\partial p}{\partial x} &= \frac{\partial}{\partial y} (-\rho U v + \mu \zeta) \\ \text{and} \qquad +\frac{\partial p}{\partial y} &= \frac{\partial}{\partial x} (-\rho U v + \mu \zeta) \end{aligned}\right\} \quad \ldots \ldots \quad D\,(13)$$

From D (13) it is seen that p is the conjugate harmonic function to $\mu\zeta - \rho U v$. Since the boundary value of the latter is known the methods of an earlier paper† indicate a method of finding p. It is, however, more simple and direct to use the solution found to substitute for v and ξ, and then integrate to find p.

From the equation

$$\psi_M = A L_{OM} + \int L_{EM} \, d\psi_E + \sum_1^{\infty} \frac{G_n \sin nM}{OM^n}, \quad \ldots \ldots \quad D\,(14)$$

the value of the molecular rotation is found by differentiation as

$$\nabla^2 \psi_M \equiv \zeta_M = A \nabla^2 L_{OM} + \int \nabla^2_M L_{EM} \, d\chi_E \quad \ldots \ldots \quad D\,(15)$$

By use of the expressions for $\partial\lambda/\partial x$ and $\partial\lambda/\partial y$ given by A (6) and the relation between L and λ, D (15), becomes

$$\zeta_M = \frac{2}{\pi} \frac{\partial}{\partial y_M} \left\{ A e^{kx_M} K_0 (k OM) + \int e^{kx_{EM}} K_0 (k EM) \, d\chi_E \right\}. \quad \ldots \quad D\,(16)$$

* LAMB, 'Hydrodynamics,' p. 573.

† 'Roy. Soc. Proc.,' A, vol. 95 (1919).

Since $U/\nu = 2k$, equation D (13) may be written as

$$-\frac{1}{\mu}\frac{\partial p}{\partial x} = -2k\frac{\partial^2\psi}{\partial x\,\partial y} + \frac{\partial\zeta}{\partial y}$$

$$= -2k\frac{\partial^2\psi}{\partial x\,\partial y} + \frac{2}{\pi}\cdot\frac{\partial^2}{\partial y^2}\left\{\mathrm{A}e^{kx_{\mathrm{M}}}\mathrm{K}_0(k\mathrm{OM}) + \int e^{kx_{\mathrm{EM}}}\mathrm{K}_0(k\mathrm{EM})\,d\chi_{\mathrm{E}}\right\} \quad . \ \mathrm{D}\,(17)$$

The differential equation satisfied by the terms in curly brackets makes it possible to substitute for the operator $\partial^2/\partial y^2$ in terms of differentials with respect to x and D (17) becomes

$$-\frac{1}{\mu}\frac{\partial p}{\partial x} = -2k\frac{\partial^2\psi}{\partial x\,\partial y}$$

$$+\frac{4k}{\pi}\left(1-\frac{1}{2k}\frac{\partial}{\partial x}\right)\frac{\partial}{\partial x}\left\{\mathrm{A}e^{kx_{\mathrm{M}}}\mathrm{K}_0(k\mathrm{OM}) + \int e^{kx_{\mathrm{EM}}}\mathrm{K}_0(k\mathrm{EM})\,d\chi_{\mathrm{E}}\right\}. \ . \ \mathrm{D}\,(18)$$

This obviously admits of integration with respect to x leaving

$$-\frac{p}{\mu} = f(y) - 2k\frac{\partial\psi}{\partial y}$$

$$+\frac{4k}{\pi}\left(1-\frac{1}{2k}\frac{\partial}{\partial x}\right)\left\{\mathrm{A}e^{kx_{\mathrm{M}}}\mathrm{K}_0(k\mathrm{OM}) + \int e^{kx_{\mathrm{EM}}}\mathrm{K}_0(k\mathrm{EM})\,d\chi_{\mathrm{E}}\right\}. \quad . \ \mathrm{D}\,(19)$$

Dealing with D (13) in the same way leads to a second value for p/μ which contains an arbitrary function of x instead of $f(y)$. By comparison of the forms it appears that $f(y)$ is an absolute constant. Reference to the resistance-formula D (9) shows that such a term adds nothing to the integral and we may therefore write $f(y) = 0$.

Using the differential forms for λ, changes may be made in D (19) and D (16) for then

$$-\frac{p}{\mu} = 2ku + 2k\mathrm{A}\frac{\partial\lambda_{\mathrm{OM}}}{\partial y_{\mathrm{M}}} + 2k\int\frac{\partial\lambda_{\mathrm{EM}}}{\partial y_{\mathrm{M}}}\,d\chi_{\mathrm{E}}, \quad . \ . \ . \ . \ . \ . \ . \ \mathrm{D}\,(20)$$

and

$$\zeta = \qquad 2k\mathrm{A}\frac{\partial\lambda_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} + 2k\int\frac{\partial\lambda_{\mathrm{EM}}}{\partial x_{\mathrm{M}}}\,d\chi_{\mathrm{E}}, \quad . \ . \ . \ . \ . \ . \ . \ \mathrm{D}\,(21)$$

and using these in D (9) it is found that

$$\mathrm{R} = \mu\int_0^{2\pi} 2k\left\{\mathrm{A}\left(\frac{\partial\lambda_{\mathrm{OM}}}{\partial y_{\mathrm{M}}}\cos\mathrm{M} - \frac{\partial\lambda_{\mathrm{OM}}}{\partial x_{\mathrm{M}}}\sin\mathrm{M}\right) + \int\left(\frac{\partial\lambda_{\mathrm{EM}}}{\partial y_{\mathrm{M}}}\cos\mathrm{M} - \frac{\partial\lambda_{\mathrm{EM}}}{\partial x_{\mathrm{M}}}\sin\mathrm{M}\right)d\chi_{\mathrm{E}}\right\}d\mathrm{M}$$

$$= 2k\mu\int_0^{2\pi}\left\{\mathrm{A}\frac{\partial\lambda_{\mathrm{OM}}}{\partial s_{\mathrm{M}}} + \int\frac{\partial\lambda_{\mathrm{EM}}}{\partial s_{\mathrm{M}}}\,d\chi_{\mathrm{E}}\right\}d\mathrm{M}. \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ \mathrm{D}\,(22)$$

Since $ds_{\mathrm{M}} = d\mathrm{M}$, D (22) is easily integrated to give

$$\mathrm{R} = 2k\mu\left[\mathrm{A}\lambda_{\mathrm{OM}} + \int\lambda_{\mathrm{EM}}\,d\chi_{\mathrm{E}}\right]_0^{\mathrm{M}}. \ . \ . \ . \ . \ . \ . \ . \ . \ \mathrm{D}\,(23)$$

Now λ_{OM} is cyclic and changes by $\frac{1}{k\pi} \cdot 2\pi$, *i.e.*, by $2/k$, as a result of integration. If λ_{EM} be regarded as the limit when M approaches the boundary from outside then its cyclic change is also $2/k$ and the integral with respect to χ_E is zero. Hence the resistance formula takes the simple form

$$R = 4\mu A. \qquad \text{D (24)}$$

In terms of velocity, etc., as given by B (39), when $C_q = 0$,

$$R = \frac{4\pi\mu U}{K_0 I_0 + K_1 I_1}, \qquad \text{D (25)}$$

and is the value for Lamb's approximation; the form is in some respects preferable to A (18).

Generalisation of the Resistance Calculation to cover Cylinders of which the Radius is not Unity.

The principles of dynamical similarity can be used to generalise D(24), for they show that

$$\frac{Rd}{\rho U^2 d^2} = F\left(\frac{Ud}{\nu}\right), \qquad \text{D (26)}$$

where d is the diameter of the cylinder and F is an undefined function; D(24) can be written as

$$R = 4\rho U^2 \cdot \frac{\nu}{U} \cdot \frac{A}{U}. \qquad \text{D (27)}$$

Now $\nu/U = 1/2k$, and hence

$$R = \rho U^2 \cdot \frac{2}{k} \cdot \frac{A}{U} \qquad \text{D (28)}$$

is the expression found by the present calculations when $d = 2$.

If in D (26) we put $d = 2$, and rearrange the terms it is found that

$$R = \rho U^2 . \, 2F\,(2U/\nu), \qquad \text{D (29)}$$

which shows that

$$F = \frac{1}{k} \cdot \frac{A}{U}. \qquad \text{D (30)}$$

In aeronautics F is known as a "drag coefficient" and values are obtained experimentally.

5. *Comparison of Calculated and Observed Resistances on a Circular Cylinder.*

A very complete series of observations of the resistance of wires has been made by E. F. Relf* at the National Physical Laboratory using a wind channel of the type developed for aeronautical experiments. The results are plotted by him in a form

* *Loc. cit.* (*supra p.* 385).

suggested by the law of dynamical similarity, but in themselves constitute a strong argument for its validity. All the experiments were made in air under atmospheric conditions so that ν was not subject to appreciable change; on the other hand, large variations were made in the velocity of the wind and in the diameter of the wires used. The range of Ud/ν was from about 10 at one end to 20,000 at the other, representing respectively tests on cylinders 0·002 in. diameter at 10 ft./sec. and 1·25 ins. diameter at 50 ft./sec. More than 100 observations distributed over the range give a definite resistance coefficient for the cylinder at any value of Ud/ν within the range covered, and from the smooth curve through the observations have been read the values in Table II below.

TABLE II.

$\log_{10} \frac{Ud}{\nu}$.	Resistance coefficient		$\log_{10} \frac{Ud}{\nu}$.	Resistance coefficient observed.
	Calculated.	Observed.		
$\bar{1}\cdot4$	14·8	—	2·2	0·66
$\bar{1}\cdot6$	10·7	—	2·4	0·65
$\bar{1}\cdot8$	7·9	—	2·6	0·60
0	6·0	—	2·8	0·55
0·2	4·65	—	3·0	0·51
0·4	3·65	—		
0·6	2·95	—	3·2	0·48
0·8	2·44	—	3·4	0·48
1·0	2·09	1·60	3·6	0·52
			3·8	0·55
1·2	1·85	1·25		
1·4	1·67	0·98	4·0	0·57
1·6	—	0·80		
1·8	—	0·73	4·2	0·59
2·0	—	0·68	4·4	0·60

Resistance coefficient = resistance per unit length/$\rho U^2 d$, where ρ is the density of the fluid, U the relative velocity, and d the diameter of the cylinder. It is a quantity of no dimensions and therefore independent of the system of units used.

Only a small range of overlapping occurs between the calculated and observed values and for $\log_{10} \frac{Ud}{\nu} = 1\cdot0$, $1\cdot2$ and $1\cdot4$, the calculation gives values 31 per cent., 48 per cent. and 60 per cent. greater than those observed. It is, however, apparent that little error would be incurred in the estimation of resistance coefficient were the calculated and observed curves joined, as shown by the dotted line of fig. 3. For very small values of Ud/ν the formula for resistance coefficient is

$$\frac{Rd}{\rho U^2 d^2} = \frac{\pi}{k\{K_0(k)\, I_0(k) + K_1(k)\, I_1(k)\}}, \quad . \quad . \quad . \quad . \quad . \qquad D(31)$$

where

$$k = \tfrac{1}{4}\frac{Ud}{\nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad D\,(32)$$

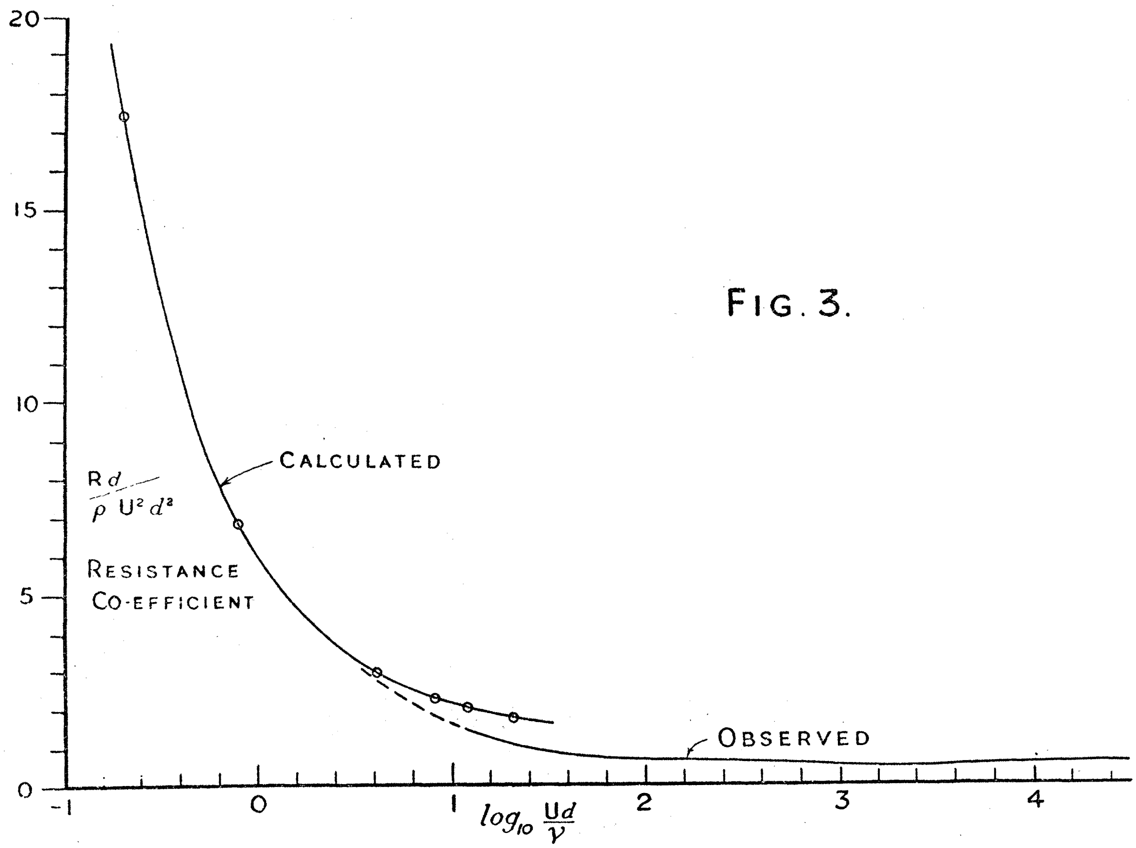


FIG. 3.

There would be little difficulty in extending the calculations to larger values of k, and for the purposes of extension of mathematical method to the more complete differential equation such solution might be a useful starting point. For the moment it would appear that the maximum use has been made of OSEEN's approximation to the equations of viscous fluid motion as applied to this particular case.

A point of interest arises in connection with the departures between calculation and observation in the range $k = 1$ to $k = 5$; it does not appear to be probable that the method of solution leaves any sensible error between the statement of the problem and the answer and the difference must then be sought in the premises. There is, of course, a limit to the accuracy of the observations, but it is not thought that any errors there are of present interest. In forming the equations of motion with the OSEEN limitation, only part of the inertia terms have been included, and these lead to a steady motion, so far as can be seen, without limitation on Ud/ν. On the other hand, the real motion of a viscous fluid round a circular cylinder becomes periodic for large values of Ud/ν. The following summary of a further paper by RELF* gives information on this point :—

* "The Singing of Circular and Stream-line Wires." By E. F. RELF, A.R.C.Sc., and E. OWER, B.Sc., A.C.G.I. Rpt., T. 1570, Aeronautical Research Committee, March, 1921.

"By means of visual observation of the eddies behind wires in water, and aural observations of the frequency of the musical note emitted when the wires were moved through air, it has been established that the singing note heard from circular wires and yawed stream-line wires has the same frequency as that of the periodic eddies produced behind the wire."

The observations indicate types of motion in which the effects of inertia become increasingly important as Ud/ν increases. Beginning with very small values, the stage of steady motion persists until Ud/ν is about 100 ($k = 25$). Periodic motion then develops which is markedly dependent on Ud/ν until a value of 500 is exceeded. The appropriate variable on the basis of dynamical similarity is $\sim d/U$ where $\sim$ is the frequency of the eddies. From $Ud/\nu = 500$ and upwards, the changes in $\sim d/U$ are of the order of errors of observation only, and taken in conjunction with the curve for resistance coefficient, the observations on singing indicate a fairly permanent régime for large value of Ud/ν.

Using the experiments as a guide it is proposed to attempt further progress by replacing the inertia terms at present omitted from the differential equation and solving for the steady motion indicated for values of k less than 25. The examination of periodic motion presents further difficulties as it probably involves examination of the stability of steady motion. It may be that the decreasing convergency shown by our present solution as k increases is a preliminary indication of the critical nature of the steady motion.

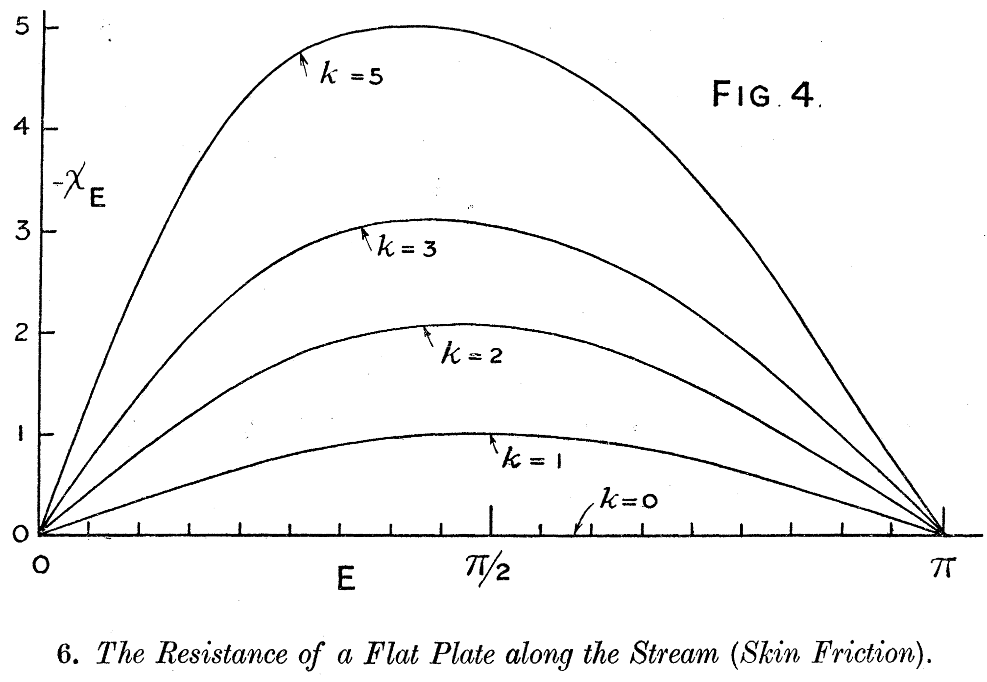
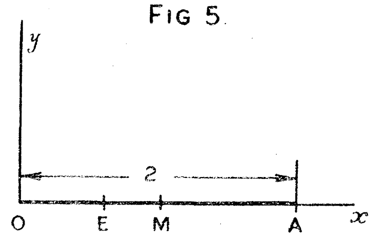


FIG. 4.

6. *The Resistance of a Flat Plate along the Stream* (*Skin Friction*).

The problem of finding the motion of fluid about a plane in the direction of the undisturbed stream admits of solution by a variation of method; it is in fact a somewhat simpler problem than that of the circular cylinder. It is convenient to regard the problem as one in which the fluid is at rest at infinity and the plate is then moving in its own plane.

FIG 5.



The plate is taken as lying in the plane xoz and of infinite length in the direction of the z axis. The origin of co-ordinates is taken at the leading edge of the plane. From experience with the previous example on the circular cylinder, we assume that

$$\psi_{\mathrm{M}} = \mathrm{AL}_{\mathrm{OM}} + \int \mathrm{L}_{\mathrm{EM}}\, d\chi_{\mathrm{E}} + \mathrm{H}_{\mathrm{M}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{E}\,(1)$$

is an appropriate expression for the stream function. In this formula A is a constant, L is a function already defined, χ_{E} represents a distribution of sources on the plate, whilst H_{M} is a harmonic function which vanishes at infinity. On the plate the fluid must be moving with the plate and the boundary conditions are

$$\left.\overline{\frac{\partial\psi}{\partial y_{\mathrm{M}}}}\right|_b = \mathrm{U}, \qquad \left.\overline{\frac{\partial\psi}{\partial x_{\mathrm{M}}}}\right|_b = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{E}\,(2)$$

where $-\mathrm{U}$ is the uniform velocity of the plate in its own plane.

From considerations of symmetry, it is clear that $\psi_{\mathrm{M}} = 0$ all along the axis of x and since L_{OM} and L_{EM} are then zero, it follows that $\mathrm{H}_{\mathrm{M}} = 0$ on the boundary; since H_{M} also vanishes at infinity it must be zero everywhere. The condition that $\left.\overline{\frac{\partial\psi}{\partial x_{\mathrm{M}}}}\right|_b = 0$ is then automatically satisfied for all values of A and χ_{E}.

From the definitions for L given earlier and reproduced in slightly different form below it appears that $\partial\mathrm{L}_{\mathrm{OM}}/\partial y_{\mathrm{M}}$ becomes infinite when $\mathrm{OM} = 0$, and since infinite velocities are not admissible it follows that A must be zero. The problem may now be stated more precisely as follows :—

In terms of the polar co-ordinates, r, θ, and the Cartesian co-ordinates x, y the function L and its first derivatives are given by

$$\pi k\mathrm{L} = kr\,\mathrm{K}_0(kr)\int_0^\theta \{e^{kr\cos\alpha}\cos\alpha - \mathrm{I}_1(kr)\}\,d\alpha + kr\,\mathrm{K}_1(kr)\int_0^\theta \{e^{kr\cos\alpha} - \mathrm{I}_0(kr)\}\,d\alpha, \quad . \qquad \mathrm{E}\,(3)$$

$$\frac{\partial\mathrm{L}}{\partial x} = \frac{1}{\pi k}\frac{\partial}{\partial y}\{e^{kx}\,\mathrm{K}_0(kr) + \log kr\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{E}\,(4)$$

$$\frac{\partial\mathrm{L}}{\partial y} = \frac{2}{\pi}e^{kx}\,\mathrm{K}_0(kr) - \frac{1}{\pi k}\cdot\frac{\partial}{\partial x}\{e^{kx}\,\mathrm{K}_0(kr) + \log kr\}. \quad . \quad . \quad . \quad . \quad . \qquad \mathrm{E}\,(5)$$

K_n and I_n are the regular Bessel functions tabulated under that head in the 'Treatise on Bessel Functions,' by GRAY, MATHEWS and MACROBERT.

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The stream function

$$\psi_M = \int L_{EM}\, d\chi_E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(6)$$

satisfies the equation of motion of a viscous fluid in the form

$$\nabla^2 \zeta_M = \frac{U}{\nu} \cdot \frac{\partial \zeta}{\partial x_M} \equiv 2k \frac{\partial \zeta}{\partial x_M}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(7)$$

which has been proposed by OSEEN as appropriate to low velocities and an infinite expanse of fluid. ζ is the molecular rotation defined by

$$\zeta_M = \nabla^2 \cdot \psi_M . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(8)$$

The solution is completed if χ_E can be found such that

$$\overline{\frac{\partial \psi}{\partial y_M}}\bigg|_b = U, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(9)$$

or by substitution from E (6) if χ_E can be determined to satisfy the equation

$$U = \int \frac{\partial L_{EM}}{\partial y_M} d\chi_E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(10)$$

when M is a point on the plane. In this case, $r = EM$ is the difference $x_M \sim x_E$, and an examination of E (5) will show that $\frac{\partial L_{EM}}{\partial y_M}$ is a function of $k(x_M - x_E)$ only. If we take χ_E as being a function of kx_E, an assumption which is, of course, quite general, equation E (10) may be re-written

$$U = \int_0^2 \frac{\partial L_{EM}}{\partial y_M} \chi'(kx_E)\, k dx_E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(11)$$

or writing ξ_E for kx_E

$$U = \int_0^{2k} \frac{\partial L_{EM}}{\partial y_M} \chi'(\xi_E)\, d\xi_E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(12)$$

where $\frac{\partial L_{EM}}{\partial y_M}$ is a function of $\xi_M - \xi_E$, a difference which it will be convenient to denote by w. Equation E (12) then contains k explicitly only in the upper limit of the integral. Substitute for the $\frac{\partial L_{EM}}{\partial y_M}$ of E (12) its value as deduced from E (5) to get

$$\frac{\pi U}{2} = \int_0^{2k} \left\{ e^w K_0(w) - \tfrac{1}{2} \frac{\partial}{\partial w} \left\{ \left(e^w K_0(w) + \tfrac{1}{2} \log w^2 \right) \right\} \chi'(\xi_E)\, d\xi_E \right. \quad . \quad . \quad E(13)$$

$$= -\int_{\xi}^{\xi_M - 2k} \left\{ e^w K_0 - \tfrac{1}{2} \frac{\partial}{\partial w} \left(e^w K_0 + \tfrac{1}{2} \log w^2 \right) \right\} \chi'(\xi_E)\, dw \quad . \quad . \quad . \quad E(14)$$

If $\chi'(\xi_E)$ is everywhere finite, a solution of E (14) can be obtained approximately by assuming that $\chi'(\xi_E)$ is constant between arbitrarily chosen values of ξ_E, as was indicated in an earlier paper. The method appears to be well known in the solution of integral equations. Between the limits $\xi_E = 0$ and $\xi_E = a$ let the value of $\chi'(\xi_E)$ be assumed to be s_{0a}, between $\xi_E = a$ and $\xi_E = b$ let $\chi'(\xi_E)$ be s_{ab} and so on, then E (14) becomes

$$\frac{\pi U}{2} = -\Sigma s_{ab} \int_{\xi_M - a}^{\xi_M - b} \left\{ e^w K_0 - \tfrac{1}{2} \frac{\partial}{\partial w} (e^w K_0 + \tfrac{1}{2} \log w^2) \right\} dw. \quad . \quad . \quad . \quad E(15)$$

The integration required by E (15) presents no special difficulty and it will be found that

$$\frac{\pi U}{2} = -\Sigma s_{ab} [we^w (K_0 + K_1) - \tfrac{1}{2} (e^w K_0 + \tfrac{1}{2} \log w^2)]_{\xi_M - a}^{\xi_M - b}. \quad . \quad . \quad . \quad E(16)$$

For any one value of ξ_M the numerical coefficients of s_{ab}, etc., can readily be computed from E (16) giving an equation of the type

$$-\frac{\pi U}{2} = \beta_1 s_{oa} + \beta_2 s_{ab} + \ldots \beta_n s_{p2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(17)$$

By taking n values of ξ_M it is clear that a system of simultaneous equations may be built up which will suffice for the determination of the value of s_{ab}, etc. The choice of the intervals ab is of some importance, and the best arrangement is often not found until after one or more preliminary solutions.

Solution when $k = 1$.—The width of the plate has been taken as 2, corresponding with a diameter of 2 for the circular cylinder. It has been shown in reference to the earlier problem that the essential variable on the principles of dynamical similarity is the ratio $\frac{U \times OA}{\nu}$, *i.e.*, $2k \times OA$. Variations in the motion can, in consequence of this knowledge, be represented by variations of k and in the case where $OA = 2$ we have

$$\frac{U \times OA}{\nu} = 4k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(18)$$

The values for ξ_M which were chosen in forming equations of the type shown in E (17) were

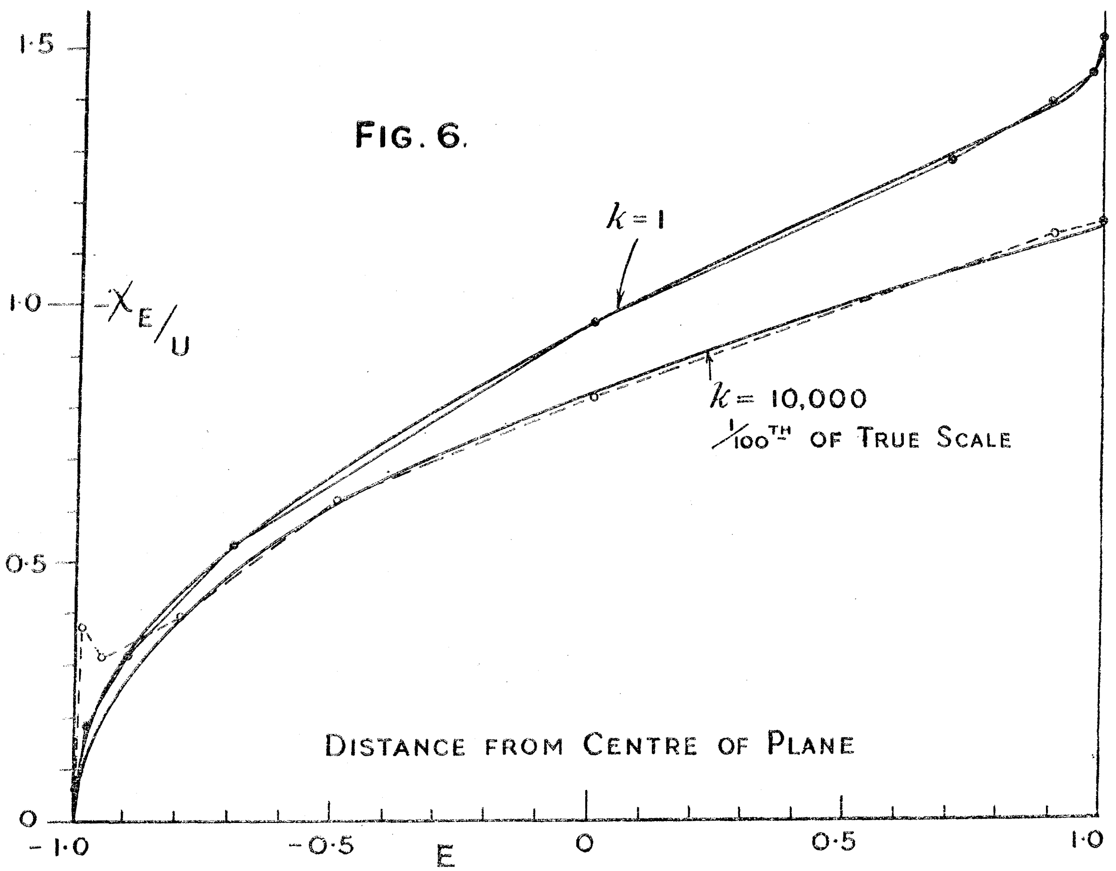
$$0,\ 0 \cdot 1,\ 0 \cdot 2,\ 0 \cdot 6,\ 1 \cdot 4,\ 1 \cdot 8,\ 1 \cdot 9,\ 2 \cdot 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad E(19)$$

whilst the corresponding sections of χ_E and the values of that quantity itself are indicated in the following table :—

TABLE III.

ξ_E.	$-\chi_E/U$.	ξ_E.	$-\chi_E/U$.
0·0	0·0	1·70	1·285
0·02	0·185	1·90	1·393
0·10	0·320	1·98	1·453
0·30	0·533	2·00	1·527
1·00	0·967		

Since $k = 1$ it follows that $\xi_E = x_E$ in this instance. From the values tabulated $-\chi_E/U$ has been plotted in fig. 6, where the angular points are marked on the upper curve by dots. In addition to the polygon the figure shows a smooth curve going through the points, which would give an adequate representation of χ_E. It is, of course, necessary that equation E (10) be satisfied for all values of M and not merely at a number of isolated points, and the possibility of drawing a continuous curve devoid of rapid reversals of slope has come to be recognised by us as an essential element of a satisfactory solution on the lines now attempted.



The indication given by this solution that χ'_E becomes infinite at the edges of the plate had not been anticipated, nor does lack of recognition of this fact appear to have led to appreciable error in the case where k was equal to unity. On the other hand, the attempted extension to $k = 10,000$ led to difficulties on this account, which will now be dealt with. The result of proceeding as for $k = 1$, with x_M equal to

$$0,\ 0\cdot5,\ 0\cdot10,\ 0\cdot3,\ 0\cdot6,\ 1\cdot0,\ 1\cdot4,\ 2\cdot0, \ldots \qquad \text{E (20)}$$

is shown by the dotted curve of fig. 6, whilst the smooth curve which satisfied equation E (10) to a sufficient degree of approximation is also shown. In the course of several variations in the choice of angular points and of values of ξ_M it was found that the form of the curve for χ_E was not uncertain except near the leading edge of the plate, and in particular the total interval in χ_E between the front edge and the trailing edge was remarkably constant. Of the four specific cases attempted the variation from O to A did not exceed 0·5 per cent. It will be found later that the whole change in χ_E is the only quantity involved in the estimation of the resistance of the plate. For the estimation of resistance it appears that errors of the type shown in fig. 6 are unimportant. In order to complete the analysis, however, it is necessary to consider what variation of method is required to deal with infinite values of χ'_E. It is clear that near the ends O and A no assumption of finite slope can suffice, but it is not equally clear what change can be made to deal with the section beginning at O and with that ending at A.

In looking for the law of variation of χ_E in the neighbourhood of the edges of the plate it was noted that it depended primarily on the existence of the singularity in $\frac{\partial L_{EM}}{\partial y_M}$ this quantity being infinite when E is at M. The singularity may be isolated by re-writing equation E (13) as

$$\frac{\pi U}{2} = \int_0^{2k}\left(1-\tfrac{1}{2}\frac{\partial}{\partial w}\right)(e^w K_0+\tfrac{1}{2}\log w^2)\,\chi'_E\,d\xi_E-\tfrac{1}{2}\int_0^{2k}\log w^2\cdot\chi'_E\,d\xi_E. \qquad \text{E (21)}$$

Since χ'_E is infinite when $\xi_E = 0$, whilst χ_E is finite, it follows that χ'_E tends to infinity in the same way as ξ^{-p} where $p < 1$.

Re-write equation E (21) as

$$T(\xi_M) \equiv -\frac{\pi U}{2} + \int_0^{2k}\left(1-\tfrac{1}{2}\frac{\partial}{\partial w}\right)(e^w K_0+\tfrac{1}{2}\log w^2)\,\chi'_E\,d\xi_E$$

$$-\tfrac{1}{2}\int_{\xi_M}^{2k}\log w^2\chi'_E\,d\xi_E = \int_0^{\xi_M}\log(\xi_M-\xi_E)\,\chi'(\xi_E)\,d\xi_E. \qquad \text{E (22)}$$

$T(\xi_M)$ is finite and has a finite first differential coefficient. We are particularly interested in the nature of the variation of $T(\xi_M)$ when ξ_M is small. By simple transformations equation E (22) can be given the form

$$T(\xi_M) = \log\xi_M\{\chi(\xi_M)-\chi(0)\} + \int_0^1\log(1-s)\,\chi'(\xi_M\,.\,s)\,\xi_M\,ds. \qquad \text{E (23)}$$

Now the integral

$$\int_0^1\log(1-s)\,\phi'(\xi_M s)\,\xi_M\,ds = -\frac{\pi^2}{6}\quad\text{if}\quad \phi'(\xi_M s) = \frac{1}{\xi_M s}, \qquad \text{E (24)}$$

and since $\chi'(\xi_M s) < \frac{1}{\xi_M s}$ as indicated earlier it follows that the integral of equation E (23) represents a finite function of ξ_M which tends to zero as ξ_M tends to zero. At the same limit it will not generally be the case that $T(\xi_M)$ will vanish and from E (23) we then deduce the relation

$$\chi(\xi_M)-\chi(0) = \frac{T(0)}{\log\xi_M}. \qquad \text{E (25)}$$

To incorporate this deduction into the method of solution one may proceed as before until equation E (14) is reached and then take account of the fact that $\chi_E = B/\log \xi_M$ when ξ_M is small, and $\chi_E = \dfrac{C}{\log(2k-\xi_M)}$ when ξ_M is nearly equal to $2k$, B and C being regarded as undetermined multipliers to be found subsequently. Equation E (16) will then become

$$\frac{\pi U}{2} = -\sum_{a}^{2-a} s_{ab}\Big[we^w(K_0+K_1)-\tfrac{1}{2}(e^wK_0+\tfrac{1}{2}\log w^2)\Big]_{\xi_M-a}^{\xi_M-b}$$
$$+B\int_0^a\left\{e^wK_0-\tfrac{1}{2}\frac{\partial}{\partial w}(e^wK_0+\tfrac{1}{2}\log w^2)\right\}\frac{\partial}{\partial\xi_E}\left(\frac{1}{\log\xi_E}\right)d\xi_E$$
$$+C\int_{2k-a}^{2k}\left\{e^wK_0-\tfrac{1}{2}\frac{\partial}{\partial w}(e^wK_0+\tfrac{1}{2}\log w^2)\right\}\frac{\partial}{\partial\xi_E}\left(\frac{1}{\log(2k-\xi_E)}\right)d\xi_E, \quad . \quad E\,(26)$$

where α is some small quantity arbitrarily chosen. The integrals of E (26) are not determinable so simply as those of E (15) but numerical evaluation does not present any formidable difficulties. Equation E (17) would follow as before except that two new constants B and C have been introduced instead of β_1 and β_n.

The full curve of fig. 6 for $k = 10{,}000$ was completed in the neighbourhood of the leading edge of the plate by a process of trial and error which proved to be rapid and effective.

Incidentally, the process indicated a connection between the values of χ_E for different values of k when k is large. The function $\partial L_{EM}/\partial y_M$ varies very rapidly in the neighbourhood of M and tends to zero as EM gets numerically great. The function however decreases much more rapidly when E is to the right of M than when to the left. When k is large—much less than 10,000—the value of the integral of E (12) is not sensibly dependent on the upper limit until ξ_M is nearly equal to $2k$. From this it follows that $\chi'(\xi_E)$ is sensibly independent of k from the leading edge to near the trailing edge so long as k is large. We may, therefore, deduce the value of χ_E in the front half of a plate when $k = 20{,}000$ from that when $k = 10{,}000$ and so on.

It follows from such considerations that the change of χ_E from the front to the back of the plate will increase with k but less rapidly than in proportion to k. When the resistance is estimated in a later section it will be found that this deduction leads to the conclusion that the resistance coefficient of a flat plate along the stream tends to decrease progressively as k increases, but not so rapidly as $1/k$.

The Resistance of the Plate.

Since the plate is flat and in the plane xoz the resistance formula is simply written down as

$$R = -2\mu\int_0^2\left(\frac{\partial^2\psi}{\partial y_M^2}\right)_{y=0}dx_M, \quad . \quad . \quad . \quad . \quad . \quad F\,(1)$$

the number 2 taking account of the two sides of the plate. The evaluation of this integral needs some little care on account of singularities in the differential coefficients of L. In terms of χ_E as may be seen from E (6)

$$R = -2\mu \int_0^2 \left\{ \int \frac{\partial^2 L_{EM}}{\partial y_M^2} d\chi_E \right\} dx_M. \qquad F(2)$$

By direct differentiation of E (5) the value of $\frac{\partial^2 L_{EM}}{\partial y_M^2}$ is found to be zero except when EM is vanishingly small and there becomes infinite.

From the differential expressions for L it will be found that

$$\frac{\partial^2 L_{EM}}{\partial y_M^2} + \frac{\partial^2 L_{EM}}{\partial x_M^2} = \frac{2}{\pi} \frac{\partial}{\partial y_M} e^{kx_{EM}} K_0 (k\text{EM}), \qquad F(3)$$

$$= 2k \frac{\partial L_{EM}}{\partial x_M} - \frac{2}{\pi} \frac{\partial}{\partial y_M} \log k\text{EM}, \qquad F(4)$$

$$= 2k \frac{\partial L_{EM}}{\partial x_M} + \frac{2}{\pi} \frac{\partial \theta_{EM}}{\partial x_M}. \qquad F(5)$$

Equation F (2) now becomes, by change of order of integration,

$$R = -2\mu \int \left\{ \int_0^2 \left(-\frac{\partial^2 L_{EM}}{\partial x_M^2} + 2k \frac{\partial L_{EM}}{\partial x_M} + \frac{2}{\pi} \frac{\partial \theta_{EM}}{\partial x_M} \right)_{y=0} dx_M \right\} d\chi_E, \qquad F(6)$$

and F (6) admits of integration with respect to x_M.

All along the axis of x, L_{EM} is zero, and hence its two derivatives $\left(\frac{\partial^2 L_{EM}}{\partial x_M^2}\right)_{y=0}$ and $\left(\frac{\partial L_{EM}}{\partial x_M}\right)_{y=0}$ vanish, including the values at the origin.

Except when M is at E, $\frac{\partial \theta_{EM}}{\partial x_M}$ is zero, but with M to the right of E the value of θ_{EM} is zero, whilst when M is to the left of E, $\theta_{EM} = \pi$. The integral with respect to x_M is then seen to be

$$R = -4\mu \int d\chi_E, \qquad F(7)$$

$$= -4\mu \{\chi_2 - \chi_0\}. \qquad F(8)$$

An interesting fact can be deduced from the comparison of the resistance of the flat plate and cylinder, since it will be found that the resistance in each case is the same when the flow at infinity is the same.

The stream function for the flat plate being

$$\psi_M = \int L_{EM} \, d\chi_E \qquad F(9)$$

expand L_{EM} by TAYLOR's theorem to get

$$\psi_M = \int \left\{ L_{OM} + (y_{EM} - y_{OM}) \frac{\partial L_{OM}}{\partial y_M} + (x_{EM} - x_{OM}) \frac{\partial L_{OM}}{\partial x_M} + \ldots \right\} d\chi_E. \qquad F(10)$$

At infinity the derivatives of L_{OM} are zero and hence

$$(\psi_M)_\infty = L_{OM}\int d\chi_E = L_{OM}(\chi_2 - \chi_0). \quad \text{F (11)}$$

In the case of the cylinder the resistance was $-4\mu A$ and by the same process as is now followed the flow at infinity is

$$AL_{OM} \quad \text{F (12)}$$

and the theorem as to resistance and flow at infinity then follows.

The molecular rotation ζ.—Since

$$\zeta_M \equiv \nabla^2\psi_M = \int \nabla_M{}^2 L_{EM}\, d\chi_E. \quad \text{F (13)}$$

it follows that ζ_M may be obtained by differentiation from results already obtained. For some purposes it is convenient to transform equation F (13) by using the expression obtained from E (5) and E (6) that

$$\nabla^2(L) = \frac{2}{\pi}\frac{\partial}{\partial y} e^{kx} K_0(kr) = 2k\frac{\partial L}{\partial x} - \frac{2}{\pi}\frac{\partial}{\partial y}\log kr, \quad \text{F (14)}$$

for it then follows that

$$\zeta_M = 2kv_M - \frac{2}{\pi}\int \frac{\partial}{\partial y_M}\log kr\, d\chi_E \quad \text{F (15)}$$

$$= 2kv_M - \frac{2}{\pi}\int \frac{\partial \theta_{EM}}{\partial x_M} d\chi_E$$

$$= 2kv_M + \frac{2}{\pi}\int \frac{\partial \theta_{EM}}{\partial x_E} d\chi_E$$

$$= 2kv_M + \frac{2}{\pi}\int \frac{\partial \chi_E}{\partial x_E} d\theta_{EM}. \quad \text{F (16)}$$

On the plate itself v_M is zero whilst $d\theta_{EM}$ is zero except when E passes M when there is an instantaneous change of $-\pi$.

Hence on the upper surface

$$\zeta_M = -2\left(\frac{\partial \chi_E}{\partial x_E}\right)_M. \quad \text{F (17)}$$

The pressure p.—From the formula G (13) for the general case of a cylinder of any form it is readily deduced that for the flat plate, where $A = 0$

$$-p/\mu = 2ku + 2k\int \frac{\partial \lambda_{EM}}{\partial y_M} d\chi_E$$

$$= 2ku + 2k\int\left(\frac{\partial L_{EM}}{\partial y_M} + \frac{1}{k\pi}\frac{\partial \theta_{EM}}{\partial y_M}\right) d\chi_E = \frac{2}{\pi}\int \frac{\partial \theta_{EM}}{\partial y_M}\partial\chi_E$$

$$= -\frac{2}{\pi}\int \frac{\partial \log k_{EM}}{\partial x_M} d\chi_E = \frac{2}{\pi}\int \frac{\partial \chi_E}{\partial x_E} d_E \log k_{EM} \quad \text{F (18)}$$

or

$$p/\mu = \frac{1}{\pi}\int \zeta_E d_E \log kEM. \quad \text{F (19)}$$

From the values of χ_E already found values of ζ and p can be calculated from F (17) and F (19). Both functions become infinite at the edges of the plate, and this property appears to arise directly from the assumption of a plate of zero thickness. Reference to the formula given by LAMB for a small cylinder will show that p tends to become infinite as the diameter of the cylinder tends to zero. It may then be concluded that the assumption of incompressibility made at the outset of our calculations is not justified at the leading and trailing edges of an infinitely thin plate, at least in conjunction with OSEEN'S approximation. The region affected by this new consideration is probably very limited in area, and it may be expected that with this exception the solution is satisfactory.

Resistance coefficient.—If d be the width of the plate, then it may be shown, as for the cylinder, that the resistance coefficient

$$k_D \equiv \frac{Rd}{\rho U^2 d^2} = \frac{1}{k}\,\frac{\chi_2 - \chi_0}{U} \qquad \text{F (20)}$$

In numerical value the present calculations lead to

$$\left.\begin{aligned} k_D &= 1 \cdot 527 \quad \text{when} \quad k = 1, \\ \text{and} \qquad k_D &= 0 \cdot 01164 \quad \text{when} \quad k = 10{,}000, \end{aligned}\right\} \qquad \text{F (21)}$$

whilst it has already been mentioned that k_D is falling as k increases but less rapidly than $1/k$.

7. *Comparison of Calculated and Observed Resistances*.

It is not possible to make an accurate comparison with experiment for the reason that observation has not been extended to cover two-dimensional flow. Experiments were made by FROUDE on the resistance of boards towed through water, but the greatest dimension was in the direction of motion, whilst the dimension across the stream was much smaller. In air ZAHM made similar experiments, and STANTON has shown that both sets are in general agreement. The resistance coefficient does in fact fall continuously with increase in k. The values of k are much greater than the 10,000 used in our present calculations.

For the purposes of comparison, use will be made of a special experiment by STANTON.* The plate was thin, 18 ins. long in the direction of the stream and 3 ins. across it, so that two-dimensional flow cannot be assumed to a high degree of approximation. The surface was ground flat, and the leading and back edges were bevelled off to a knife-edge on one side, so that the eddy-making at these edges should be a minimum. Attempts were made to determine the intensity of the skin friction, for which part of the method seems particularly suitable, but the flow was found to be unsymmetrical on the two sides of the plate and further experiment was deferred. There is evidence of the

* Advisory Committee for Aeronautics, R. and M., No. 631, February, 1919.

existence of a small eddy from the leading edge of the plate in a reversal of the pressure measured. The observations made show quite clearly that the changes of pressure at the leading edge are extremely great, but were not sufficient to determine values very close to the knife-edge.

For the present purpose it is proposed only that comparison between observation and calculation be made on the resistance coefficients. STANTON states that the resistance of the plate was 1,160 dynes at a wind speed of 30 feet per second and 3,470 dynes at 50 ft. per second. The value of the kinematic viscosity would be 0·000159 sq. ft. per second, whilst the density was 0·00237 slugs per cubic ft.

From these observations may be deduced the results

$$\left.\begin{aligned} k_D &= 0\cdot0033 \quad \text{when} \quad k = 70{,}000 \\ \text{and} \qquad k_D &= 0\cdot0035 \quad \text{when} \quad k = 120{,}000 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad F(22)$$

as found by direct measurement. The values of k_D are rather less than one-third of that calculated for $k = 10{,}000$, whilst the average value of k in the experiment is roughly ten times as great as that in the calculation.

In view of the uncertainties of the comparison it has not been thought profitable to pursue the calculations further, but it is clear that the resistance coefficient observed can be predicted approximately by the methods of calculation here followed. It may be that, apart from the small eddy due to lack of symmetry, the motion of a real fluid over a flat plate is steady for the values of k reached in experiment.

8. *Solution for a Cylinder of any Section.*

From the particular solutions obtained in the cases of the circular cylinder and the flat plate it appears probable that the stream function in the general case can be expressed in the form

$$\psi_M = AL_{OM} + \int L_{EM}\, d\chi_E + H_M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad G(1)$$

where H_M is a solution of LAPLACE's equation which vanishes at infinity.

The solution of equations of the type shown by G (1) was required in our earlier work, and an account of the processes followed is given on pp. 397–409, 'Roy. Soc. Proc.,' A, vol. 100, 1922. The present procedure is a further application of the same idea.

In fig. 7 the closed curve represents the section of the cylinder and is arbitrary; it may for example represent the section of an aeroplane wing or strut. F is a point lying on this boundary and M a point in the viscous fluid; its position relative to F is defined by the polar co-ordinates FM and θ_{FM}. In applying GREEN's theorem it is essential to specify the direction of the normal to the boundary, and this also is shown in fig. 7.

With these definitions the appropriate form of GREEN's theorem shows that

$$2\pi H_M = -\int_{\text{circuit}} H_F \, d\theta_{FM} + \int \frac{\partial H_F}{\partial n_F} \log FM \, ds_F, \quad . \quad . \quad . \quad . \quad . \qquad G(2)$$

and there is no restriction on the expanse of fluid since H vanishes at infinity. Applying G (2) to G (1) shows that

$$2\pi\left(\psi_M - AL_{OM} - \int L_{EM}\, d\chi_E\right) = -\int_{\text{circuit}} \left(\psi_F - AL_{OF} - \int L_{EF}\, d\chi_E\right) d\theta_{FM}$$
$$+ \int \left(\frac{\partial \psi}{\partial n_F} - A\frac{\partial L_{OF}}{\partial n_F} - \int \frac{\partial L_{EF}}{\partial n_F}\, d\chi_E\right) \log FM \, ds_F, \quad . \qquad G(3)$$

and G (3) is an integral equation for χ_E in terms of the boundary conditions. The

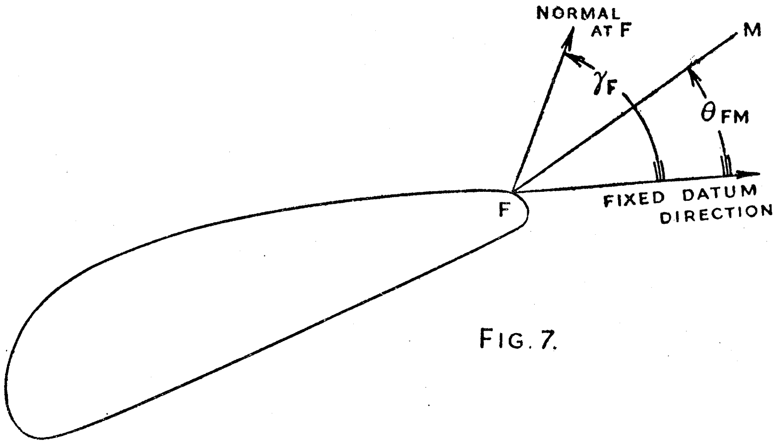


FIG. 7.

latter may be expressed simply, in the case of zero slipping at the boundary. An axis of x is taken parallel to the direction of motion of the cylinder and then

$$(\psi_M)_b = U\,(y_{OM})_b, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad G(4)$$

where the agreement also extends to the first derivatives with respect to x and y. The choice of the origin O is arbitrary on theoretical grounds, but when integrals are being evaluated by graphical methods, a suitable location may save labour in working to a given degree of accuracy.

With M on the boundary the only unknown in G (3) is χ_E. Re-arrange the equation as

$$\pi\,(\psi_M - AL_{OM}) + \int_{\text{tangent}}^{\text{tangent}} (\psi_F - AL_{OF})\, d\theta_{FM} - \int \left(\frac{\partial \psi}{\partial n_F} - A\frac{\partial L_{OF}}{\partial n_F}\right) \log FM \, ds_F$$
$$= \int \left(\pi L_{EM} + \int_{\text{tangent}}^{\text{tangent}} L_{EF}\, d\theta_{FM} - \int \frac{\partial L_{EF}}{\partial n_F} \log FM \, ds_F\right) d\chi_E, \quad . \qquad G(5)$$

and re-write the equation as

$$\varpi_M = \int Y_{EM}\, d\chi_E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad G(6)$$

where it should be noted, ϖ_M and Y_{EM} are completely defined functions of the positions of the two boundary points E and M.

The approximate method of solution followed for the flat plate can be applied to G (6). Divide the perimeter of the section of the cylinder into parts, not necessarily of equal size, and in each interval assume constancy for $\frac{\partial \chi_E}{\partial s_E}$. The number of parts t which is satisfactory will depend on the complexity of the problem and the degree of accuracy required. Let the various values of $\frac{\partial \chi_E}{\partial s_E}$ be denoted by $C_1, C_2 \ldots C_t$ and let a particular position of M be indicated by "A." Equation G(6) becomes

$$\varpi_a = C_1 \int_1 Y_{Ea}\, ds_E + C_2 \int_2 Y_{Ea}\, ds_E \ldots C_t \int_t Y_{Ea}\, ds_E, \quad . \quad . \quad . \quad . \quad . \quad \text{G}(7)$$

the suffixes to the integrals representing a limit in the range of s which corresponds with the limitations in the C^s. The integrals are all completely determinate in numerical form and leave an equation which is linear in $C_1, C_2 \ldots C_t$.

By taking M at some other point a further equation is obtained, and finally the number of simultaneous equations becomes sufficient for the determination of $C_1, C_2 \ldots C_t$. This is, of course, the essential solution for χ_E.

Some degree of skill, acquired only by experience, appears to be required in applying this process. The choice of the divisions and the positions of the location points "A" offer an infinite number of possible arrangements, some of which are much more favourable than others. At best the process is tentative, and we have usually started with a series of 6 or 8 equations from which to deduce χ_E and completed the calculations. Some two sections show a greater change in their C^s than others, and if the difference is very great the two sections are replaced by three. There is no reason to regard the method as uncertain if treated in this way.

Resistance and lift.—The calculation of resistance follows the lines of that given for the cylinder. Instead of the angle M we use γ_M, the angle which the normal at M makes with the axis of x. It is also necessary to consider forces normal to the direction of motion. The definitions of the pressure components are

$$\left.\begin{aligned} p_{yy} &= -p+2\mu\frac{\partial v}{\partial y}, \\ p_{xx} &= -p+2\mu\frac{\partial u}{\partial x}, \\ p_{xy} &= \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{G}(8)$$

whilst the force components are

$$R_x = \int p_{xx} \cos\gamma_M\, ds_M + \int p_{xy} \sin\gamma_M\, ds_M \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{G}(9)$$

and

$$R_y = \int p_{yy} \sin\gamma_M\, ds_M + \int p_{xy} \cos\gamma_M\, ds_M . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{G}(10)$$

By the same processes as were followed for the circular cylinder equations G (9) and G (10) can be shown to become

$$\mathrm{R}_x = -\int (p \cos \gamma_{\mathrm{M}} + \mu\zeta \sin \gamma_{\mathrm{M}})\, ds_{\mathrm{M}}, \quad \text{G (11)}$$

$$\mathrm{R}_y = -\int (p \sin \gamma_{\mathrm{M}} - \mu\zeta \cos \gamma_{\mathrm{M}})\, ds_{\mathrm{M}}, \quad \text{G (12)}$$

whilst the expressions for p and ζ are unchanged from the forms D (20) and D (21).

The expressions for p and ζ in terms of λ are

$$-p/\mu = 2ku + 2k\mathrm{A}\frac{\partial\lambda_{\mathrm{OM}}}{\partial y_{\mathrm{M}}} + 2k\int \frac{\partial\lambda_{\mathrm{EM}}}{\partial y_{\mathrm{M}}}\, d\chi_{\mathrm{E}}. \quad \text{G (13)}$$

and

$$\zeta = 2k\mathrm{A}\frac{\partial\lambda_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} + 2k\int \frac{\partial\lambda_{\mathrm{EM}}}{\partial x_{\mathrm{M}}}\, d\chi_{\mathrm{E}}. \quad \text{G (14)}$$

From which is obtained

$$\mathrm{R}_x = 2k\mu \int \Big[-\mathrm{U} \cos \gamma_{\mathrm{M}} + \mathrm{A}\left(\frac{\partial\lambda_{\mathrm{OM}}}{\partial y_{\mathrm{M}}} \cos \gamma_{\mathrm{M}} - \frac{\partial\lambda_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} \sin \gamma_{\mathrm{M}}\right)$$
$$+ \int \left\{ \frac{\partial\lambda_{\mathrm{EM}}}{\partial y_{\mathrm{M}}} \cos \gamma_{\mathrm{M}} - \frac{\partial\lambda_{\mathrm{EM}}}{\partial x_{\mathrm{M}}} \sin \gamma_{\mathrm{M}} \right\} d\chi_{\mathrm{E}} \Big]\, ds_{\mathrm{M}} \quad \text{G (15)}$$

$$= -2k\mu\mathrm{U}\int dy_{\mathrm{M}} + 2k\mu \int \left[\mathrm{A}\frac{\partial\lambda_{\mathrm{OM}}}{\partial s_{\mathrm{M}}} + \int \frac{\partial\lambda_{\mathrm{EM}}}{\partial s_{\mathrm{M}}}\, d\chi_{\mathrm{E}} \right] ds_{\mathrm{M}} \quad \text{G (16)}$$

$$= 4\mu\mathrm{A} \quad \text{G (17)}$$

This generalises the formula for resistance already found for the circular cylinder and flat plate.

Similarly

$$\mathrm{R}_y = 2k\mu \int \Big[-\mathrm{U} \sin \gamma_{\mathrm{M}} + \mathrm{A}\left\{\frac{\partial\lambda_{\mathrm{OM}}}{\partial y_{\mathrm{M}}} \sin \gamma_{\mathrm{M}} + \frac{\partial\lambda_{\mathrm{OM}}}{\partial x_{\mathrm{M}}} \cos \gamma_{\mathrm{M}}\right\}$$
$$+ \int \left\{ \frac{\partial\lambda_{\mathrm{EM}}}{\partial y_{\mathrm{M}}} \sin \gamma_{\mathrm{M}} + \frac{\partial\lambda_{\mathrm{EM}}}{\partial x_{\mathrm{M}}} \cos \gamma_{\mathrm{M}} \right\} d\chi_{\mathrm{E}} \Big]\, ds_{\mathrm{M}}. \quad \text{G (18)}$$

From the differential relations for λ_{EM} it can be seen that

$$\frac{\partial\lambda_{\mathrm{EM}}}{\partial y_{\mathrm{M}}} \sin \gamma_{\mathrm{M}} + \frac{\partial\lambda_{\mathrm{EM}}}{\partial x_{\mathrm{M}}} \cos \gamma_{\mathrm{M}} = \frac{2}{\pi}\, e^{kx_{\mathrm{EM}}}\mathrm{K}_0(k\mathrm{EM}) \sin \gamma_{\mathrm{M}} - \frac{1}{\pi k}\frac{\partial}{\partial s_{\mathrm{M}}}\, e^{kx_{\mathrm{EM}}}\mathrm{K}_0(k\mathrm{EM}), \quad \text{G (19)}$$

and that R_y becomes

$$\mathrm{R}_y = -\frac{4k\mu}{\pi}\int_{\text{contour}} \left\{ \mathrm{A}e^{kx_{\mathrm{M}}}\mathrm{K}_0(k\mathrm{OM}) + \int e^{kx_{\mathrm{EM}}}\mathrm{K}_0(k\mathrm{EM})\, d\chi_{\mathrm{E}} \right\} ds_{\mathrm{M}}. \quad \text{G (20)}$$

Except in the case of symmetry it is not obvious that R_y will vanish, but rather that a lift may be expected.

APPENDIX.

Calculation of the Values of λ *and* L.

In much of the preceding work it has been sufficient to work with the differential coefficients of λ and L, but in the more general cases it is probable that values of λ and L will be required in addition to those of their differentials.

λ has been defined in equation A (6), p. **387**, as

$$\frac{\partial \lambda}{\partial x} = \frac{1}{k\pi}\frac{\partial}{\partial y} e^{kx} \mathrm{K}_0(kr), \qquad \text{H (1)}$$

$$\frac{\partial \lambda}{\partial y} = \frac{2}{\pi}\left(1 - \frac{1}{2k}\frac{\partial}{\partial x}\right) e^{kx} \mathrm{K}_0(kr), \qquad \text{H (2)}$$

and it is now proposed to develop the integral for λ.

Change the variables to

$$z = kx, \quad a = ky, \quad \xi = kr, \quad \theta = \tan^{-1} a/z, \qquad \text{H (3)}$$

with the differentials

$$\frac{\partial \xi}{\partial a} = \frac{a}{\xi}, \quad \frac{\partial \xi}{\partial z} = \frac{z}{\xi}, \quad \frac{\partial \theta}{\partial a} = \frac{z}{\xi^2}, \quad \frac{\partial \theta}{\partial z} = -\frac{a}{\xi^2}. \qquad \text{H (4)}$$

With these variables H (1) and H (2) become

$$\pi k \frac{\partial \lambda}{\partial a} = e^{\xi \cos\theta}\left\{\mathrm{K}_0(\xi) - \cos\theta \frac{\partial \mathrm{K}_0(\xi)}{\partial \xi}\right\} \qquad \text{H (5)}$$

and

$$\pi k \frac{\partial \lambda}{\partial z} = \sin\theta e^{\xi\cos\theta} \frac{\partial \mathrm{K}_0(\xi)}{\partial \xi}. \qquad \text{H (6)}$$

Assume for $\pi k \lambda(\xi, \theta)$ a relation of the type

$$\pi k \lambda = e^{\xi\cos\theta} \cdot \phi(\xi, \theta), \qquad \text{H (7)}$$

from which by differentiation we find

$$\pi k \frac{\partial \lambda}{\partial a} = e^{\xi\cos\theta}\left\{\frac{\partial \phi}{\partial \xi}\sin\theta + \frac{1}{\xi}\cdot\frac{\partial \phi}{\partial \theta}\cos\theta\right\} \qquad \text{H (8)}$$

and

$$\pi k \frac{\partial \lambda}{\partial z} = e^{\xi\cos\theta}\left\{\phi + \frac{\partial \phi}{\partial \xi}\cos\theta - \frac{1}{\xi}\cdot\frac{\partial \phi}{\partial \theta}\sin\theta\right\}. \qquad \text{H (9)}$$

Combining H (5) and H (8), and H (6) and H (9) leads to the elimination of the factor $e^{\xi\cos\theta}$ and leaves two equations for ϕ. They are

$$\frac{\partial \phi}{\partial \xi}\sin\theta + \frac{\cos\theta}{\xi}\frac{\partial \phi}{\partial \theta} = \mathrm{K}_0 - \cos\theta\frac{\partial \mathrm{K}_0}{\partial \xi} \qquad \text{H (10)}$$

and

$$\phi + \frac{\partial \phi}{\partial \xi}\cos\theta - \frac{\sin\theta}{\xi}\frac{\partial \phi}{\partial \theta} = \sin\theta\frac{\partial \mathrm{K}_0}{\partial \xi}. \qquad \text{H (11)}$$

Multiply H (10) by **sin** θ and H (11) by cos θ and add to get

$$\phi \cos \theta + \frac{\partial \phi}{\partial \xi} = \mathrm{K}_0 \sin \theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(12)$$

and by a similar process

$$-\phi \sin \theta + \frac{1}{\xi} \cdot \frac{\partial \phi}{\partial \theta} = \mathrm{K}_0 \cos \theta - \frac{\partial \mathrm{K}_0}{\partial \xi}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(13)$$

Equation H (13) may be integrated in the usual way, the integrating factor being $e^{\xi \cos \theta}$. The result is

$$e^{\xi \cos \theta} \cdot \phi = \int_0^\theta \xi e^{\xi \cos \alpha} \left\{ \mathrm{K}_0 \cos \alpha - \frac{\partial \mathrm{K}_0}{\partial \xi} \right\} d\alpha + f(\xi). \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(14)$$

Differentials from H (14) may be used in H (12) to determine the value of $f(\xi)$, and remembering that

$$\frac{\partial^2 \mathrm{K}_0}{\partial \xi^2} + \frac{1}{\xi} \frac{\partial \mathrm{K}_0}{\partial \xi} = \mathrm{K}_0, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(15)$$

it is found that $f'(\xi) = 0$, and $f(\xi)$ a constant independent of ξ or θ. There is no loss of generality in the problems concerned if $f(\xi)$ be made zero. Finally, therefore, we have

$$\pi k \lambda = \int_0^\theta \xi e^{\xi \cos \alpha} \left\{ \mathrm{K}_0 \cos \alpha - \frac{\partial \mathrm{K}_0}{\partial \xi} \right\} d\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(16)$$

Using Sonine's expansion for $e^{\xi \cos \alpha}$, viz. :—

$$e^{\xi \cos \alpha} = \mathrm{I}_0(\xi) + 2 \sum_{n=1}^{\infty} \mathrm{I}_n(\xi) \cos n\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(17)$$

and integrating term by term leads to the result

$$\pi k \lambda = \xi \theta (\mathrm{K}_0 \mathrm{I}_1 + \mathrm{K}_1 \mathrm{I}_0) + \xi \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} \{ \mathrm{K}_0 (\mathrm{I}_{n-1} + \mathrm{I}_{n+1}) + 2\mathrm{K}_1 \mathrm{I}_n \} \quad . \quad . \quad \mathrm{H}\,(18)$$

$$= \theta + \xi \sum_{n=1}^{\infty} \frac{\sin n\theta}{n} \{ \mathrm{K}_0 (\mathrm{I}_{n-1} + \mathrm{I}_{n+1}) + 2\mathrm{K}_1 \mathrm{I}_n \}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(19)$$

H (19) follows from H (18) by the theorem in Bessel functions that

$$\mathrm{K}_0 \mathrm{I}_1 + \mathrm{K}_1 \mathrm{I}_0 = 1/\xi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(20)$$

Equation H (19) shows that λ is a cyclic function in respect of the term θ only, and the general nature of the Bessel functions involved shows the series to be convenient when ξ is small. Tables of K_0, K_1, and I_n are given at the end of the recently issued book on Bessel functions by Gray, Mathews and MacRobert.

When ξ is large a different method is more convenient. Equation H (16) may be written as

$$\pi k \lambda = \xi \mathrm{K}_1 \int_0^\theta e^{\xi \cos \alpha}\, d\alpha + \xi \mathrm{K}_0 \frac{\partial}{\partial \xi} \int_0^\theta e^{\xi \cos \alpha}\, d\alpha \quad . \quad . \quad . \quad . \quad \mathrm{H}\,(21)$$

and evaluation turns essentially round the possibility of a general estimation of

$$\int_0^\theta e^{\xi \cos \alpha}\, d\alpha.$$

Now

$$\int_0^\theta e^{\xi \cos \alpha}\, d\alpha = \int_0^\theta e^{\xi - \xi(1-\cos\alpha)}\, d\alpha = e^\xi \int_0^\theta e^{-2\xi \sin^2 \alpha/2}\, d\alpha. \quad . \quad . \quad . \quad . \quad \text{H}(22)$$

Let

$$\beta \equiv \sqrt{2\xi} \sin \alpha/2 \quad \text{and} \quad \beta_1 \equiv \sqrt{2\xi} \sin \theta/2 \quad . \quad . \quad . \quad . \quad \text{H}(23)$$

and H (22) leads to

$$\int_0^\theta e^{\xi \cos \alpha}\, d\alpha = e^\xi \sqrt{\frac{2}{\xi}} \int_0^{\beta_1} e^{-\beta^2} \left(1 - \frac{\beta^2}{2\xi}\right)^{-\frac{1}{2}} d\beta. \quad . \quad . \quad . \quad . \quad \text{H}(24)$$

By differentiation of H (24) with respect to ξ and the use of H (21) it is found that

$$\pi k\lambda = (\text{K}_0 + \text{K}_1)\, e^\xi \sqrt{2\xi} \int_0^{\beta_1} e^{-\beta^2} \left(1 - \frac{\beta^2}{2\xi}\right)^{-\frac{1}{2}} d\beta$$

$$- \text{K}_0 e^\xi \frac{1}{\sqrt{2\xi}} \int_0^{\beta_1} e^{-\beta^2} \left(1 - \frac{\beta^2}{2\xi}\right)^{-\frac{3}{2}} d\beta + \text{K}_0 \tan\frac{\theta}{2} e^\xi e^{-\beta_1^2}. \quad . \quad \text{H}(25)$$

Now use the asymptotic expansions for K_0 and K_1 which lead to

$$\frac{e^\xi}{\sqrt{\xi}} \text{K}_0 = \frac{1}{\xi} \sqrt{\frac{\pi}{2}} \left\{1 - \frac{1^2}{8\xi} + \frac{1^2 \,.\, 3^2}{2!} \cdot \frac{1}{(8\xi)^2} \cdots \right\}. \quad . \quad . \quad . \quad . \quad \text{H}(26)$$

and

$$\xi\, (\text{K}_0 + \text{K}_1) \frac{e^\xi}{\sqrt{\xi}} = \sqrt{\frac{\pi}{2}} \left\{2 - \frac{1^2 + \overline{1^2 - 4}}{8\xi} + \frac{1^2 \,.\, 3^2 + (1^2 - 4)(3^2 - 4)}{2!\,(8\xi)^2} \cdots \right\} \quad . \quad \text{H}(27)$$

Further, expand the expressions $\left(1 - \frac{\beta^2}{2\xi}\right)^{-\frac{1}{2}}$ and $\left(1 - \frac{\beta^2}{2\xi}\right)^{-\frac{3}{2}}$ by the Binomial theorem, and by use of them together with H (26) and H (27) re-write equation H (25):

$$\pi k\lambda = \sqrt{\pi} \left\{2 - \frac{1^2 + (1^2 - 4)}{8\xi}\right.$$

$$\left. + \frac{1^2 \,.\, 3^2 + (1^2-4)(3^2-4)}{2!\,(8\xi)^2} - \cdots \right\} \int_0^{\beta_1} e^{-\beta^2} \left\{1 + \sum_{n=1} \frac{1 \,.\, 3 \,.\, 5 \ldots \overline{2n-1}}{n!} \left(\frac{\beta^2}{4\xi}\right)^n \right\} d\beta$$

$$- \frac{\sqrt{\pi}}{2} \left\{1 - \frac{1^2}{8\xi} + \frac{1^2 \,.\, 3^2}{2!} \cdot \frac{1}{(8\xi)^2} - \cdots \right\} \left\{ \int_0^{\beta_1} \frac{1}{\xi} e^{-\beta^2} \left\{1 + \sum_{n=1} \frac{1 \,.\, 3 \,.\, 5 \ldots \overline{2n+1}}{n!} \left(\frac{\beta^2}{4\xi}\right)^n \right\} d\beta \right.$$

$$\left. - \sqrt{\frac{2}{\xi}} \tan\frac{\theta}{2} e^{-\beta_1^2} \right\} \quad . \quad . \quad \text{H}(28)$$

PEARSON* tabulates values of a quantity $m_{2n}(x_1)$ which is such that

$$m_{2n}(x_1) = \frac{1}{1 \,.\, 3 \,.\, 5 \ldots (2n-1)} \frac{1}{\sqrt{2\pi}} \int_0^{x_1} x^{2n} e^{-\frac{1}{2}x^2}\, dx. \quad . \quad . \quad . \quad . \quad \text{H}(29)$$

* 'Tables for Statisticians and Biometricians,' by KARL PEARSON, F.R.S.

To bring this into a form of present utility let $x^2 = 2\beta^2$ and re-write H (29)

$$m_{2n}(x_1) = \frac{1}{1\,.\,3\,.\,5\,\ldots\,(2n-1)}\,\frac{2^n}{\sqrt{\pi}}\int_0^{\beta_1} e^{-\beta^2}\beta^{2n}\,d\beta, \quad . \quad . \quad . \quad . \quad \text{H}\,(30)$$

and hence

$$\frac{1}{\sqrt{\pi}}\int_0^{\beta_1}\left(\frac{\beta^2}{4\xi}\right)^n e^{-\beta^2}\,d\beta = \{1\,.\,3\,.\,5\,\ldots\,\overline{2n-1}\}\left(\frac{1}{8\xi}\right)^n m_{2n}(x_1). \quad . \quad . \quad . \quad \text{H}\,(31)$$

To simplify the final writing and also to facilitate numerical work certain groupings of quantities are convenient; they are

$$\left.\begin{aligned} s_1 &\equiv 1-\frac{1^2}{8\xi}+\frac{1^2\,.\,3^2}{2!}\cdot\frac{1}{(8\xi)^2}-\ldots \\ s_2 &\equiv 1-\frac{1^2-4}{8\xi}+\frac{(1^2-4)\,(3^2-4)}{2!\,(8\xi)^2}-\ldots \\ \text{C}_n &\equiv \frac{(1\,.\,3\,.\,5\,\ldots\,\overline{2n-1})^2}{n!}\left(\frac{1}{8\xi}\right)^n \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{H}\,(32)$$

and in terms of them

$$2k\lambda = \left\{2\,(s_1+s_2)-\frac{s_1}{\xi}\right\}\{\tfrac{1}{2}\,(1+\alpha)_{x_1}-\tfrac{1}{2}\}$$
$$+\sum_{n=1}\left\{2\,(s_1+s_2)-\frac{(2n+1)\,s_1}{\xi}\right\}\text{C}_n m_{2n}(x_1)+s_1\sqrt{\frac{2}{\pi\xi}}\tan\frac{\theta}{2}\,e^{-x_1^2/2}. \quad . \quad . \quad \text{H}\,(33)$$

In using H (33) it is to be remembered that

$$x_1 = 2\sqrt{\xi}\,\sin\frac{\theta}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{H}\,(34)$$

The expression $\frac{1}{2}(1+\alpha)_{x_1}$ has here been introduced* as a convenient integral for the Gaussian probability integral in order to make use of PEARSON's tables. It is defined by

$$\tfrac{1}{2}\,(1+\alpha)_{x_1} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x_1} e^{-\frac{1}{2}x^2}\,dx, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{H}\,(35)$$

and it is readily found from this, that in the notation of this appendix

$$\frac{1}{\sqrt{\pi}}\int_0^{\beta_1} e^{-\beta^2}\,d\beta = \tfrac{1}{2}\,(1+\alpha)_{x_1}-\tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{H}\,(36)$$

Equation H (33) provides a convenient formula for the calculation of λ when H (19) is inconvenient.

* *Ibid.*, p. 2, *et seq.*

The relation between λ and L is

$$L = \lambda - \frac{\theta}{k\pi} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{H (37)}$$

and is sufficient to define L in terms ξ and θ. In many respects the acyclic function L is more convenient for use than λ with its cyclic change of $2/k$.

GRAPHICAL REPRESENTATION OF THE FUNCTION "L (ξ, θ)"

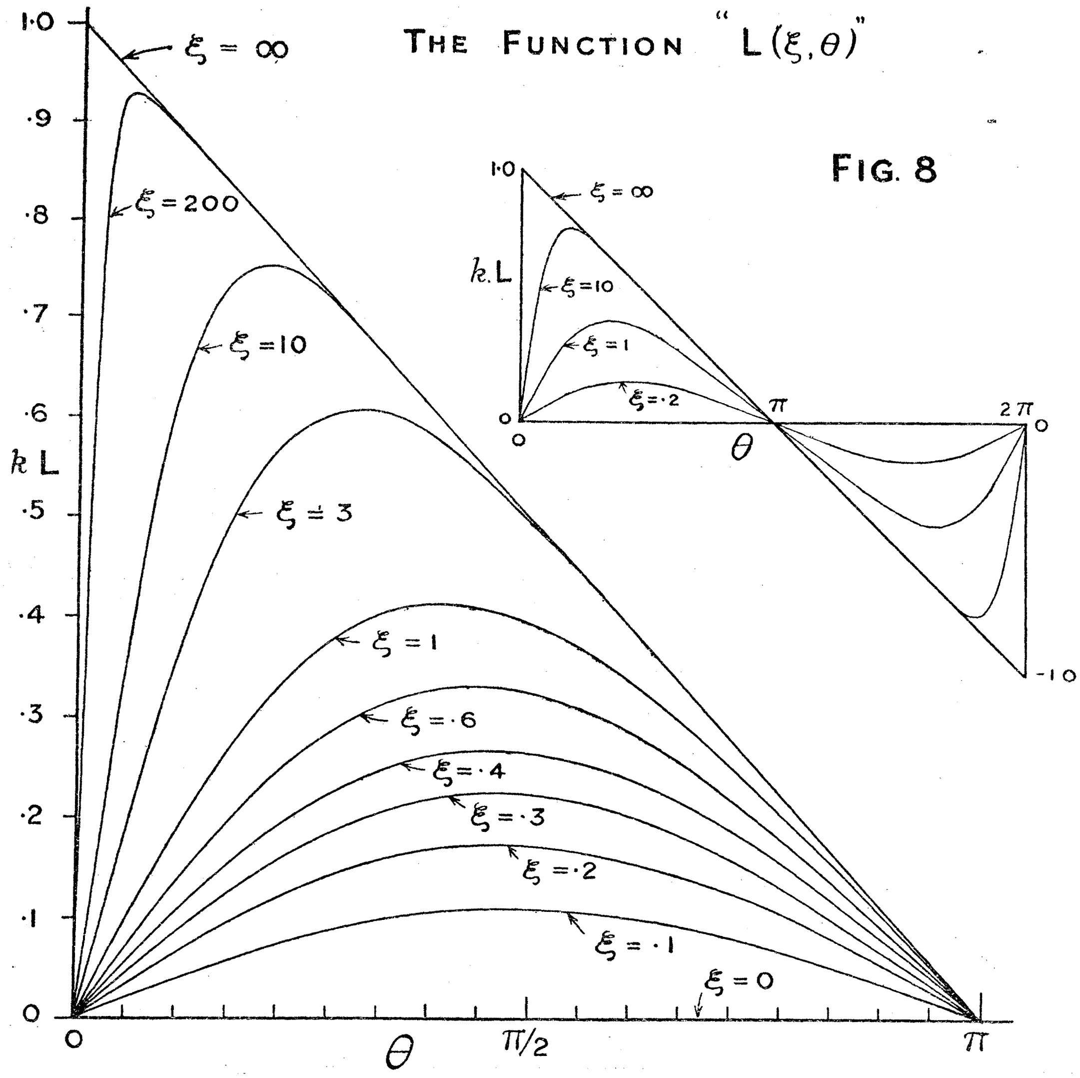


FIG. 8

As a function of θ it appears that L (ξ, θ) has an infinite differential coefficient when $\theta = 0$ and $\xi = \infty$.

[*Note added April 5th,* 1923.—Since the presentation and reading of our paper an addition to the published mathematical solutions of the flow of viscous fluids has been made by BERRY and SWAIN. The full details appear in 'Roy. Soc. Proc.,' A, vol. 102, March, 1922, p. 766, and the analysis has such a close relation to that described by us as to make a direct comparison desirable. It appears that the arbitrary constant which BERRY and SWAIN left undetermined may be deduced from the comparison, and the result is a generalization of LAMB's resistance formula for a circular cylinder to any elliptic cylinder with one of its principal axes along the direction of motion.

In order to avoid confusion of symbols we replace the k and ξ of BERRY and SWAIN by β and t, so that the elliptic co-ordinates become

$$x = \beta \cosh t \cos \eta, \qquad y = \beta \sinh t \sin \eta. \quad \text{J (1)}$$

a is the semi-axis along the stream and b that across it; in developing the formulæ, steps will only be shown for the case $a > b$, for which instance $\beta^2 = a^2 - b^2$. BERRY and SWAIN give an expression for the stream function which is

$$\frac{\psi}{2\mathrm{B}ab} = \frac{y}{\beta^2}\left\{ab + \beta^2 \tanh^{-1}\frac{b}{a} - b^2 \coth t - \beta^2 t\right\}, \quad \text{J (2)}$$

and for the vorticity $\zeta = \nabla^2\psi$

$$\frac{\zeta}{2\mathrm{B}ab} = -\nabla^2 yt = -\frac{2}{\beta}\frac{\cosh t \sin \eta}{\cosh^2 t \sin^2 \eta + \sinh^2 t \cos^2 \eta}. \quad \text{J (3)}$$

In order to compare J (2) and J (3) with our own results, we first express ψ and ζ as functions of the polar co-ordinates R and M, where

$$x = \mathrm{R} \cos \mathrm{M}, \qquad y = \mathrm{R} \sin \mathrm{M}. \quad \text{J (4)}$$

Since $\nabla^4\psi = 0$ it follows that ζ satisfies LAPLACE's equation; it is symmetrical with respect to the axis of y and asymmetrical with respect to the axis of x. Since it vanishes when R is infinite, a suitable form for ζ is

$$\frac{\zeta}{2\mathrm{B}ab} = \frac{\mathrm{A}_1 \sin \mathrm{M}}{\mathrm{R}} + \frac{\mathrm{A}_3 \sin 3\mathrm{M}}{\mathrm{R}^3} + \ldots + \frac{\mathrm{A}_{2s+1} \sin (2s+1)\mathrm{M}}{\mathrm{R}^{2s+1}} + \text{etc.} \quad \text{J (5)}$$

The values of $\mathrm{A}_1 \ldots \mathrm{A}_{2s+1}$ can be obtained most simply by considering the special case in which $\mathrm{M} = \pi/2$. J (5) then becomes

$$\left(\frac{\zeta}{2\mathrm{B}ab}\right)_{\pi/2} = \sum_1 (-1)^s \frac{\mathrm{A}_{2s+1}}{y^{2s+1}}, \quad \text{J (6)}$$

whilst J (3) reduces to

$$\left(\frac{\zeta}{2\mathrm{B}ab}\right)_{\pi/2} = \left(-\frac{2}{\beta \cosh t}\right)_{\pi/2}, \quad \text{J (7)}$$

and from J (1), $(\beta \sinh t)_{\pi/2} = y$. Since y is arbitrary, a value may be assigned to it which is greater than β, and a comparison of J (6) and J (7) then shows that

$$(-1)^s \mathrm{A}_{2s+1} = \text{coefficient of } y^{-(2s+1)} \text{ in } -\frac{2}{y}\left(1 + \frac{\beta^2}{y^2}\right)^{-1/2}, \quad \text{J (8)}$$

from which it appears that

$$\left.\begin{aligned} &A_1 = -2, \\ s \neq 1, \quad &A_{2s+1} = -\frac{1 \,.\, 3 \,.\, 5 \ldots (2s-1)}{2^{s-1} \lfloor s} \beta^{2s}. \end{aligned}\right\} \quad \text{J (9)}$$

This completely defines ζ by the use of J (5).

Since

$$\nabla^2 yt = -\frac{\zeta}{2Bab} = -\sum_0 \frac{A_{2s+1} \sin (2s+1) M}{R^{2s+1}}, \quad \text{J (10)}$$

it is possible to find a value of yt as follows. Directly from J (10) it may be seen that

$$yt = R \sin M \log R + \sum_1 \frac{A_{2s+1}}{8s} \cdot \frac{\sin (2s+1) M}{R^{2s-1}} + \text{a harmonic function.} \quad \text{J (11)}$$

On the other hand

$$t + i\eta = \cosh^{-1}\left(\frac{x+iy}{\beta}\right) = \log \frac{R}{\beta} e^{iM} \left\{1 + \sqrt{1 - \frac{\beta^2}{R^2} e^{-2iM}}\right\} \quad \text{J (12)}$$

and it follows that t is harmonic. That part of t which does not vanish when $R = \infty$ is easily found to be $\log \frac{2R}{\beta}$, and an examination of the typical terms of the expansion for t shows that

$$t = \log \frac{2R}{\beta} + \sum_1 \frac{C_{2s} \cos 2sM}{R^{2s}}. \quad \text{J (13)}$$

From this yt is found as

$$yt = R \sin M \log \frac{2R}{\beta} + \tfrac{1}{2} \sum_1 \frac{C_{2s} \{\sin (2s+1) M - \sin (2s-1) M\}}{R^{2s-1}}, \quad \text{J (14)}$$

which by comparison with J (11) shows that

$$yt = R \sin M \log \frac{2R}{\beta} + \sum_1 \frac{A_{2s+1}}{8s} \cdot \frac{\sin (2s+1) M - \sin (2s-1) M}{R^{2s-1}}. \quad \text{J (15)}$$

Before the expansion of ψ is complete, it is necessary to deal with the further term $y \coth t$, which is a harmonic function conforming generally as to symmetry with the conditions indicated earlier for ζ. When R is infinite $\coth t$ = unity, and a suitable expansion is

$$y \coth t = R \sin M - \sum_0 \frac{Q_{2s+1} \sin (2s+1) M}{R^{2s+1}}. \quad \text{J (16)}$$

If as before in the case of A_{2s+1}, we determine Q_{2s+1}, from the special case in which $x = 0$, we get from J (7)

$$\begin{aligned} (y \coth t)_{\pi/2} &= y \sqrt{1 + \frac{\beta^2}{y^2}} \\ &= y + \tfrac{1}{2} \frac{\beta^2}{y} + \sum_2 (-1)^{s+1} \frac{1 \,.\, 3 \,.\, 5 \ldots (2s-3)}{2^s \lfloor s} \frac{\beta^{2s}}{y^{2s-1}}, \end{aligned} \quad \text{J (17)}$$

which by comparison with J (16) with $M = \pi/2$ leads to

$$Q_1 = \frac{\beta^2}{2},$$

$$s \neq 0, \qquad Q_{2s+1} = -\frac{1\,.\,3\,.\,5\,\ldots\,(2s-1)}{2^{s+1}\,\underline{|s+1}}\,\beta^{2s+2} = \frac{\beta^2}{4\,(s+1)}\,A_{2s+1}, \quad . \quad . \quad J\,(18)$$

From the separate expansions the completed form for ψ can be written down as

$$\frac{\psi}{2Bab} = R \sin M \left\{\frac{b\,(a-b)}{\beta^2} + \tanh^{-1}\frac{b}{a} - \log\frac{2R}{\beta}\right\}$$
$$+ b^2 \sum_1 \frac{A_{2s+1}}{4\,(s+1)}\,\frac{\sin(2s+1)\,M}{R^{2s+1}} - \frac{a^2}{2}\,\frac{\sin M}{R}$$
$$- \sum_1 \frac{A_{2s+1}}{8s}\,\frac{\sin(2s+1)\,M - \sin(2s-1)\,M}{R^{2s-1}}, \quad . \quad . \quad . \quad . \quad . \quad J\,(19)$$

this expression being deduced wholly from BERRY and SWAIN.

The equivalent expression arising from the solution of the equations of motion in the form proposed by OSEEN will now be obtained. Using the formulæ in the body of our paper and the boundary conditions adopted by BERRY and SWAIN, we find for the stream function an expression of the form

$$\psi_M = AL_{OM} + \int L_{EM}\,d\chi_E + H_3 - Uy_M, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad J\,(20)$$

where U is the magnitude of the steady streaming, H_3 is a harmonic function which vanishes at infinity, A is a constant and χ_E a boundary function. Near the cylinder and for small values of REYNOLDS' number it has been shown that the ψ of J (20) is a solution of the equation

$$\nabla^4 \psi_M = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad J\,(21)$$

and ψ_M is there necessarily identical with the ψ of BERRY and SWAIN.

The appropriate approximation to J (20) to conform to J (21) is found by writing

$$L = \frac{y}{\pi}\left(1 - \gamma - \log\frac{kR}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad J\,(22)$$

and therefrom

$$\psi_M = \frac{A}{\pi}\,y_M\left(1-\gamma-\log\frac{kR}{2}\right) + \frac{1}{\pi}\int (y_M - y_E)\left(1-\gamma-\log\frac{kEM}{2}\right) d\chi_E + H_3 - Uy_M. \quad . \quad J\,(23)$$

From the form of the integral and considerations of symmetry it may be shown that

$$\psi = \frac{A}{\pi}\left(1-\gamma-\log\frac{kR}{2}\right) R \sin M - \frac{1}{\pi}\int (y_M - y_E) \log EM\,d\chi_E + H_3 - UR \sin M. \quad . \quad . \quad J\,(24)$$

It appears that the integral in J (24) will vanish when $R = \infty$—it will be evaluated in due course—but it may now be noted that the results so far obtained suffice to determine the coefficient B of the BERRY and SWAIN formula. From the equality of the coefficients of R sin M log R in J (19) and J (24) it follows that

$$2Bab = \frac{A}{\pi}, \qquad \text{J (25)}$$

whilst the similar equation from the coefficients of R sin M is

$$\frac{A}{\pi}\left(1-\gamma-\log\frac{k}{2}\right)-2Bab\left(\frac{b\,(a-b)}{\beta^2}+\tanh^{-1}\frac{b}{a}-\log\frac{2}{\beta}\right)=U, \qquad \text{J (26)}$$

or after a few simple transformations

$$\frac{A}{\pi}\left(\frac{a}{a+b}-\gamma-\log\frac{k\,(a+b)}{4}\right)=U. \qquad \text{J (27)}$$

Resistance Formula.

The resistance per unit length of cylinder is given by BERRY and SWAIN as

$$R = 4\pi\mu\,(2Bab),$$

and this by use of J (25) becomes

$$R = 4\mu A,$$

the formula given by us for the general case of slow motion in two dimensions as based on OSEEN'S approximation to the equations of motion. Putting in the value of A found in J (27) leads to the result

$$R = \frac{4\pi\mu U}{\dfrac{a}{a+b}-\gamma+\log\dfrac{k\,(a+b)}{4}}. \qquad \text{J (28)}$$

The resistance formula J (28) was deduced from calculations based on $a > b$, the flow being along the major axis of the ellipse. A repetition of the calculations for $b > a$ shows the formula to be unchanged if the "a" axis be along the stream in the latter case. When $a = b$ the elliptic cylinder becomes circular and J (28) identical with the formula given by LAMB.*

Comparison of the resistances of ellipses of different eccentricities, one of the principal axes being along the stream.

The only logical basis which has suggested itself to us is the calculation of the size of cylinders as a function of eccentricity, other quantities being maintained

* 'Hydrodynamics,' LAMB, 4th Edn., p. 606.

unchanged. It is supposed that the cylinders are moving in a specified fluid at a given speed, and are of such size that the resistance is constant from cylinder to cylinder. Equation J (28) may be rewritten as

$$\frac{R}{4\pi\mu U} = \frac{1}{\frac{a}{a+b} - \gamma - \log\frac{U}{8\nu} - \log(a+b)} \quad \text{J (29)}$$

For the conditions postulated for the comparison R, μ and U are constants, and from J (29) it then appears that

$$\frac{a}{a+b} - \log(a+b) = \text{constant} \equiv \log\Omega, \quad \text{J (30)}$$

or separating the terms depending on size from those which change with eccentricity only

$$\log\Omega a = \frac{1}{1+\frac{b}{a}} - \log\left(1+\frac{b}{a}\right). \quad \text{J (31)}$$

The relative dimensions of elliptic cylinders of given eccentricities and common resistance are independent of Ω, that is, of the absolute amount of the resistance. From J (31), Table A has been prepared.

TABLE A.

b/a.	a/b.	Ωa.	Ωb.
0·0		2·72	
0·2		1·92	
0·4		1·46	
0·6		1·17	
0·8		0·97	
1·0	1·0	0·82	0·82
	0·8		0·87
	0·6		0·91
	0·4		0·95
	0·2		0·99
	0·0		1·00

The result of the comparison given above is rather curious in that it places the circular cylinder in the position of being more highly resistant than a lamina one diameter wide across the stream. A plane along the stream has a maximum dimension e times as great as a plane across the stream when producing the same resistance.

The proportions indicated by Table A for small values of REYNOLDS' number differ markedly from proportions found experimentally at larger values. Experimental evidence indicates a rough agreement in the diameter of a circular cylinder and the width of a lamina across the stream when the two are giving the same resistance. A lamina along the stream would require to be some 50 to 100 times as wide as a lamina of equal resistance across the stream if the value of REYNOLDS' number is as high as that reached in experiment.

An indication of the numerical value of REYNOLDS' number to which present calculations apply was given in the early part of this paper.

Completion of the analysis of the fluid motion on the basis of J (20).

Equation J (20) represents the slow motion of a viscous fluid not only near the cylinder, but at all parts of the fluid. It contains two quantities χ_E and H_3, which have not yet been determined. A method of finding both for a cylinder of any form has been given in our paper, but in the case of the elliptic cylinder their values may be determined from the comparison of J (20) with the formula of BERRY and SWAIN.

To determine χ_E use may conveniently be made of the vorticity for

$$\zeta_M = A\nabla^2 L_{OM} + \int \nabla_M{}^2 L_{EM}\, d\chi_E. \quad \text{J (32)}$$

The approximate form of J (32) which is suitable for use near the cylinder is

$$\frac{\pi\zeta}{A} = -2\frac{\sin M}{R} - 2\int \frac{y_M - y_E}{EM^2}\, d\frac{\chi_E}{A}. \quad \text{J (33)}$$

Using the normal device for determining the coefficients of a FOURIER'S series leads to the equation

$$\frac{A_{2s+1}}{R^{2s+1}} = -\frac{1}{\pi}\int_0^{2\pi}\left\{\sin(2s+1)M \int \frac{y_M - y_E}{EM^2}\, d\frac{\chi_E}{A}\right\} dM, \quad \text{J (34)}$$

changing the order of integration and writing

$$x_E = r\cos\theta, \qquad y_E = r\sin\theta,$$

converts J (34) to

$$\frac{A_{2s+1}}{R^{2s+1}} = -\frac{1}{\pi}\int\left\{\int_0^{2\pi}\frac{(R\sin M - r\sin\theta)\sin(2s+1)M}{R^2+r^2-2Rr\cos(M-\theta)}\, dM\right\}\frac{d\chi_E}{A}. \quad \text{J (35)}$$

The denominator in the integral with respect to M may be expanded in cosines of multiple angles and the integrations performed, with the result that

$$A_{2s+1} = -\int r^{2s}\cos 2s\theta\, d\frac{\chi_E}{A}. \quad \text{J (36)}$$

When proceeding to evaluate the remaining integral it is convenient to express χ_E as a FOURIER series in η. Since E is a point on the elliptic boundary

$$r\cos\theta = a\cos\eta, \qquad r\sin\theta = b\sin\eta,$$

and

$$A_{2s+1} = \text{real part of } -\int (r\cos\theta + ir\sin\theta)^{2s}\,\frac{d\chi_E}{A},$$

i.e. of

$$-\int\left\{\frac{a+b}{2}(\cos\eta + i\sin\eta) + \frac{a-b}{2}(\cos\eta - i\sin\eta)\right\}^{2s}\frac{d\chi_E}{A} \quad . \quad . \quad . \quad \text{J (37)}$$

$$= -\int \sum_0^s \frac{\lfloor 2s}{2^{2s}\lfloor q\,\lfloor 2s-q}\left\{(a-b)^{2s-q}(a+b)^q + (a+b)^{2s-q}(a-b)^q\right\}\cos(2s-2q)\,\eta\,\frac{d\chi_E}{A}. \quad . \quad \text{J (38)}$$

Only cosines of even multiples of η in the expansion for $d\chi_E$ will lead to terms in the integral and we therefore assume that

$$d\,\frac{\chi_E}{A} = \sum_{p=1} N_{2p}\cos 2p\eta\,d\eta. \quad . \quad . \quad . \quad \text{J (39)}$$

After a number of simple changes, it is found that

$$A_{2s+1} = -\pi\sum_1^s\left(\frac{a^2-b^2}{4}\right)^s\frac{\lfloor 2s}{\lfloor s+p\,\lfloor s-p}\left\{\left(\frac{a-b}{a+b}\right)^p + \left(\frac{a+b}{a-b}\right)^p\right\}N_{2p}. \quad . \quad . \quad \text{J (40)}$$

It is possible to solve J (40) in general terms ; to see this, let

$$N'_{2p} = \frac{\pi}{4}(-1)^p\left\{\left(\frac{a-b}{a+b}\right)^p + \left(\frac{a+b}{a-b}\right)^p\right\}N_{2p} - 1, \quad . \quad . \quad . \quad \text{J (41)}$$

combining J (9) with J (40) then produces the relation

$$2\sum_1^s\frac{\lfloor 2s}{\lfloor s+p\,\lfloor s-p}(-1)^{p-1}N'_{2p} = 2\sum_1^s\frac{\lfloor 2s}{\lfloor s+p\,\lfloor s-p}(-1)^p + \frac{\lfloor 2s}{\lfloor s\,\lfloor s}$$

$$= (-1)^s\sum_0^{2s}\frac{\lfloor 2s}{\lfloor r\,\lfloor 2s-r}(-1)^r = 0. \quad . \quad . \quad . \quad \text{J (42)}$$

It follows from J (42) that N'_{2p} must be zero for all values of p, and J (41) then leads to

$$N_{2p} = \frac{4}{\pi}\,\frac{(-1)^p}{\left(\frac{a-b}{a+b}\right)^p + \left(\frac{a+b}{a-b}\right)^p} \quad . \quad . \quad . \quad \text{J (43)}$$

and with J (39) completely defines χ_E.

Determination of H_3.

From a comparison of forms J (19) and J (24) it will be found that

$$\frac{\pi}{A} H_3 = \sum_1 A_{2s+1} \left\{ \frac{b^2 \sin(2s+1)M}{4(s+1)R^{2s+1}} - \frac{\sin(2s-1)M}{8sR^{2s-1}} \right\} - \frac{a^2}{2} \frac{\sin M}{R} - \frac{A_3 \sin M}{8R}$$

$$+ \text{ harmonic part of } \int (y_M - y_E) \log EM \frac{d\chi_E}{A} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad J\,(44)$$

$$= \sum_1 \left[A_{2s+1} \left\{ \frac{b^2 \sin(2s+1)M}{4(s+1)R^{2s+1}} + \frac{\sin(2s-1)M}{8sR^{2s-1}} \right\} + \frac{\sin(2s+1)M}{4(2s+1)R^{2s+1}} \times \right.$$

$$\left. \int \left\{ r^{2s+2} \cos 2s\theta - \frac{1}{2s+2} r^{2s+2} \cos(2s+2)\theta \right\} \frac{d\chi_E}{A} \right] - \frac{a^2}{2} \frac{\sin M}{R} - \frac{A_3 \sin M}{8R}. \qquad J\,(45)$$

Making use of J (36) and the relation

$$r^2 = \frac{a^2+b^2}{2} + \frac{a^2-b^2}{2} \cos 2\eta$$

leads to the evaluation of the integral of J (45). Finally, then,

$$\frac{\pi}{A} H_3 = -\frac{a^2 \sin M}{2R} + \sum_1 \frac{\sin(2s+1)M}{R^{2s+1}} \left[\frac{A_{2s+1}}{2} \left\{ \frac{b^2}{2s+2} - \frac{a^2+b^2}{4(2s+1)} \right\} + \frac{sA_{2s+3}}{2(2s+1)(2s+2)} \right.$$

$$\left. + \frac{1}{4(2s+1)} \left(\frac{a^2-b^2}{4} \right)^{s+1} \sum_{p=0}^{s} \frac{\lfloor 2s}{\lfloor s-p \, \lfloor s+p} \left\{ \left(\frac{a-b}{a+b} \right)^p + \left(\frac{a+b}{a-b} \right)^p \right\} \{N_{2p+2} + N_{2p-2}\} \right]. \quad J\,(46)$$

The values of χ_E and H_3 having been found in general terms define the fluid motion at all points by means of J (20). No attempt has here been made to evaluate analytically the integral of J (20) at points in the fluid for which the BERRY and SWAIN formula is inapplicable. In any particular instance the graphical evaluation presents little difficulty.

It may be noted that χ_E vanishes when $a = b$, *i.e.* for the circular cylinder. This might have been demonstrated, as in our earlier paragraphs, as a consequence of the restriction of the differential equation to $\nabla^4 \psi = 0$. This approximate form excludes "inertia" terms altogether, whereas the OSEEN form of equation includes an important part of them. The extension of solutions to higher values of REYNOLDS' number with which we have dealt in the body of the paper is therefore not here applied to the more general case of the elliptic cylinder; the method of treatment for a cylinder of any shape has been indicated elsewhere, but no applications have yet been completed.]